20th Scientific-Professional Colloquium on Geometry and Graphics
Fužine, September 3 – 7, 2017

# ABSTRACTS

EDITORS:
Tomislav Došlić, Ema Jurkin

PUBLISHER:
Croatian Society for Geometry and Graphics

Sponsors:
Adriatic Sailing Ltd
CADCOM d.o.o.
SAND d.o.o.
State Geodetic Directorate of the Republic of Croatia

Supported by the Ministry of Science and Education of the Republic of Croatia and the Foundation of Croatian Academy of Sciences and Arts.

# Contents

# Plenary lectures

## Isoptic curves and surfaces

Géza Csima
Department of Geometry, IM Budapest University of Technology and Economics, Budapest, Hungary
e-mail: csgeza@math.bme.hu

Jenő Szirmai
Department of Geometry, IM Budapest University of Technology and Economics, Budapest, Hungary
e-mail: szirmai@math.bme.hu

In our talk, we would like to go through the main results of my PhD thesis. A history of the isoptic curves goes back to the 19th century, but nowadays the topic is experiencing a renaissance, providing numerous new results.

First, we define the notion of isoptic curve and outline some of the well known results for strictly convex, closed curves. The formulas of the isoptic curves to conic sections are well known since 1837, but we give a new approach to prove them. We also consider polygons, and we give a new algorithm, which provides us implicit equations of isoptic curves. This algorithm is appropriate not only for polygons but for finite point sets as well.

Choosing the solid angle from numerous spatial angle definitions, we extend our investigation to the Euclidean space $\mathbf{E}^3$. After describing a procedure for calculating solid angles, we sketch an application for the isoptic surface of rectangles (see [1]). We also develop our planar algorithm for polyhedra and random spatial point sets (see [2]). Numerous figures for the isoptic surface of Platonic and Archimedean solids will be presented (see Figure 1).

Leaving the Euclidean geometries behind, we consider the conic sections of the hyperbolic plane. The literature of the hyperbolic conic section classification is huge. Now, we give them from a new aspect, respected to the duality. We also overview the generalized angle and distance formulas of the extended hyperbolic plane suggested by Vörös Cyrill. Finally, we give the isoptic curves of the hyperbolic conic sections on the extended hyperbolic plane (see [3] and Figure 2).

Similar questions could be interesting in other Thurston geometries as well. In this talk, we only show some examples of the isoptic surface of the line segment in the $\widetilde{\mathbf{SL}_2\mathbf{R}}$.

**Key words:** isoptic curves, isoptic surfaces, non-Euclidean geometry

**MSC 2010**: 51N20, 51N15, 53A35

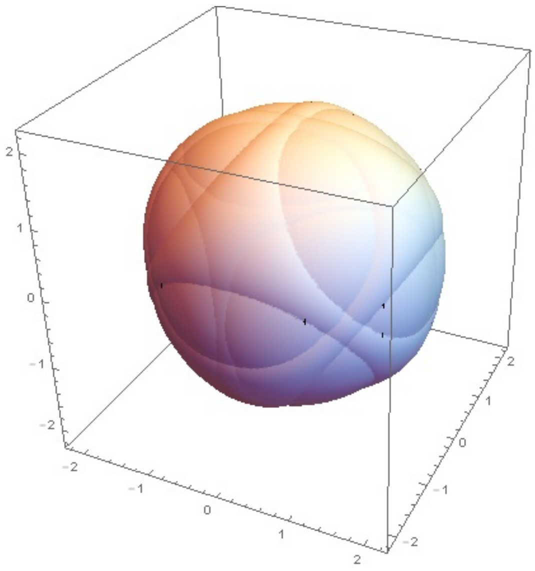

Figure 1: Isoptic surface of the truncated octahedron for $\alpha = \pi/3$

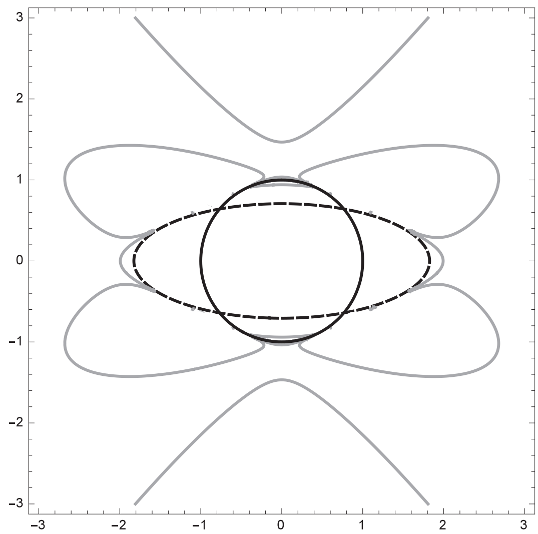

Figure 2: Isoptic curve for concave hyperbola

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# Automated Reasoning Tools for Euclidean Planar Geometry in GeoGebra

Zoltán Kovács
The Private University College of Education of the Diocese of Linz, Austria
e-mail: zoltan@geogebra.org

Computing numerical checks of certain relations between objects in a planar construction is a well known feature of dynamic geometry systems [1]. GeoGebra's [2] newest improvements offer symbolic checks of equality, parallelism, perpendicularity, collinearity, concurrency or concyclicity in a Euclidean geometry construction [3]. Also dragging of locus curves, being defined explicitly or implicitly, is a new feature to visually check a conjecture, and by getting the exact equations, actually proofs will be immediately obtained [4, 5].

These novel possibilities can be introduced in classrooms to support dynamic geometry experiments and help formulating theorems, and, what is more, scientific research can also be mediated by the computer.

During the talk some practictal introduction will be shown how these novel methods can be used to extend teaching geometry at a high school level. We will also obtain some more advanced results in Euclidean planar geometry which are not or not yet well known, by making experiments with GeoGebra's Automated Reasoning Tools.

**Key words:** automated reasoning, dynamic geometry, automatic discovery, GeoGebra, computational algebraic geometry

**MSC 2010**: 97G40, 14H50

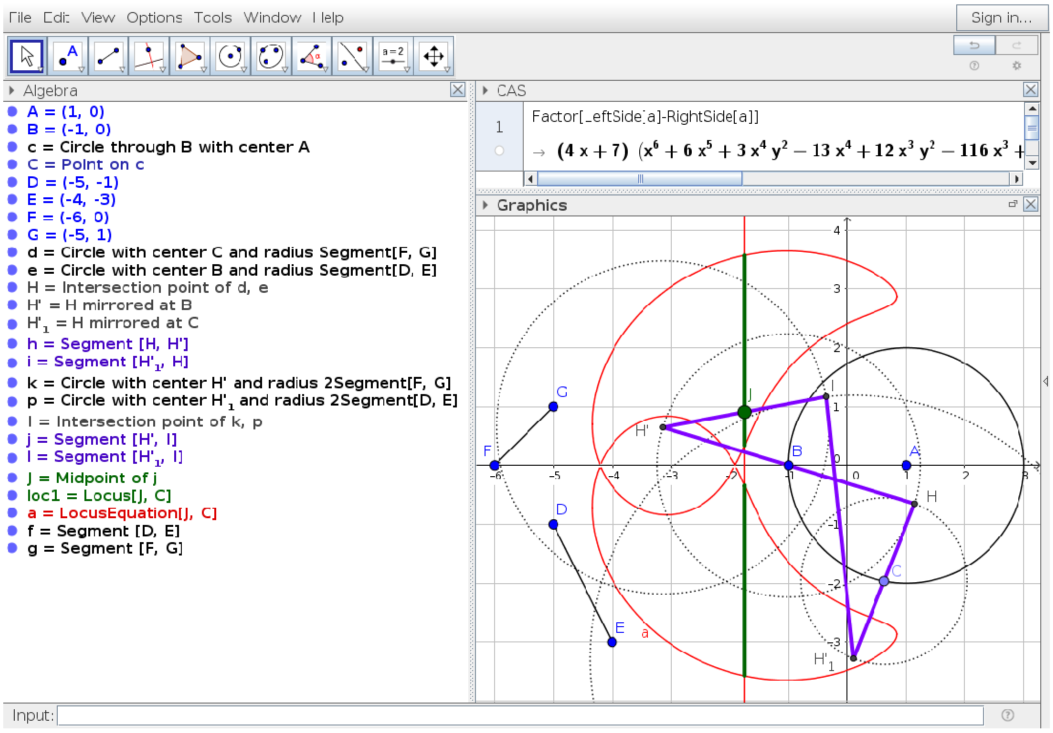


Figure 1: Hart's first inversor [6, p. 36] in GeoGebra [7]. The `LocusEquation` command obtains the exact equation of the Zariski closure [8, p. 199] of the locus curve. By using the `Factor` command it turns out that point $J$ moves on a straight line, namely on $x = -\frac{7}{4}$.

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# On Movable Single-Loop Mechanisms

Martin Pfurner
Unit for Geometry and CAD, University of Innsbruck, Innsbruck, Austria
e-mail: martin.pfurner@uibk.ac.at

Single-loop mechanisms consist of $n$ links connected in series by $n$ joints to achieve a closed chain. These kind of mechanisms attracted many researchers in the past decades. One of the most famous examples was presented by Bennett [1] in 1903 and a very good introduction and overview on mechanisms with four, five or six revolute joints can be found in the habilitation thesis of Dietmair [2], the most recent attempts were made using a different method in [3, 4].

The journey in this presentation will start with paradoxical movable mechanisms with four, five or six links.

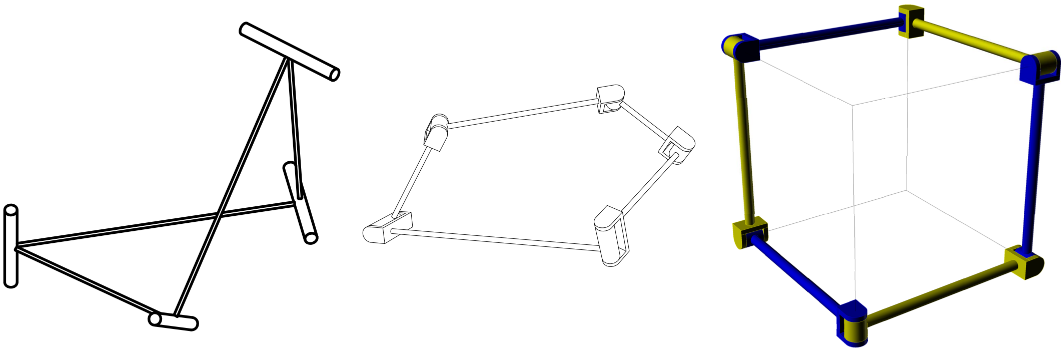

Figure 1: A Bennett, a Goldberg and an orthogonal Bricard mechanism

Using kinematic mapping synthesis and analysis problems can be solved. It is possible to find restrictions for the design such that the mechanism becomes movable. This mapping can also be used to study relations between the joint parameters (input-output equations), to analyze point paths and coupler ruled surfaces of the mechanisms during its motion. Additionally an outlook for possible applications on parallel mechanisms is shown.

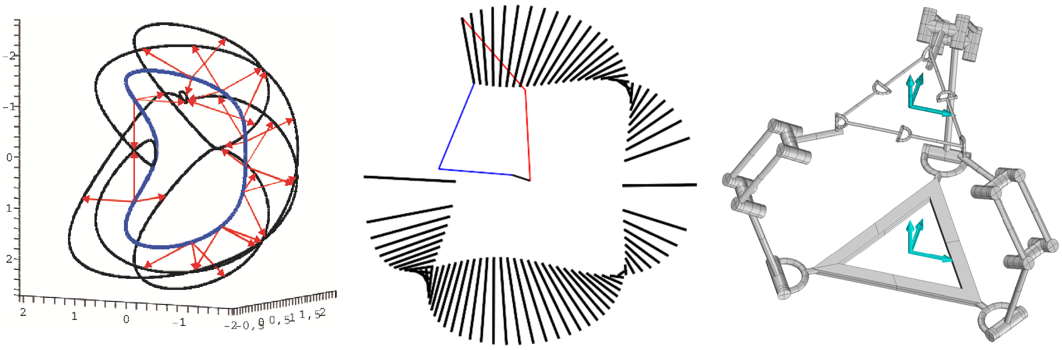

Figure 2: Point paths, discrete poses of a coupler motion and a possible application

In the second part we will extend the number of links to seven or eight and investigate properties of such mechanisms, which are always movable.

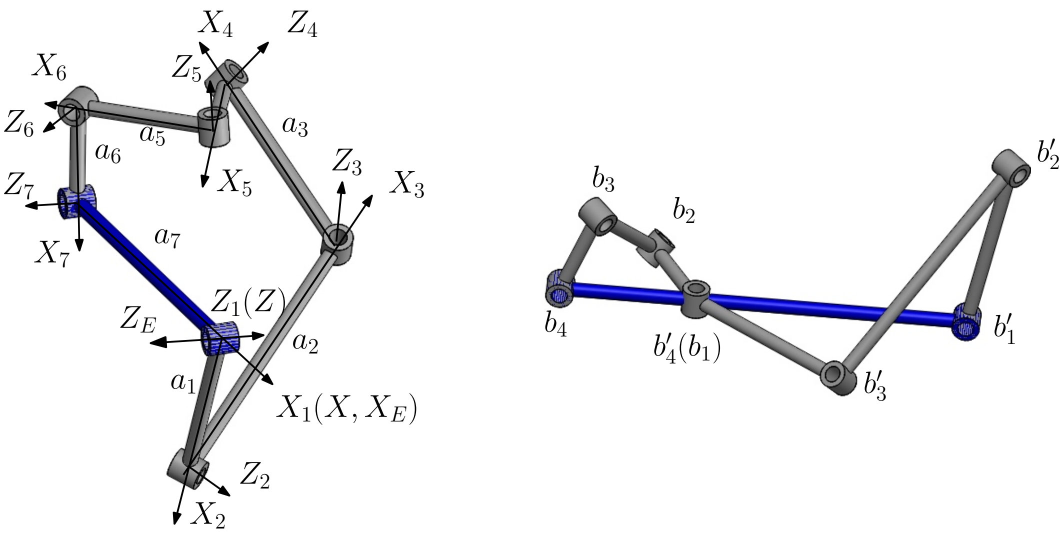


Figure 3: 7R-mechanism and one of its transition configurations

Here one can additionally ask for different motion modes with possibly different degrees of freedom and their transition configurations. The case of the 8R chain in this presentation is a concatenation of two well known 6R chains and its way of construction and analysis is shown here for the first time.

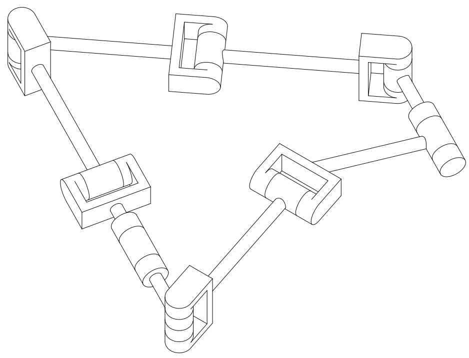

Figure 4: 8R Bricard-Schatz mechanism

**Key words:** overconstrained mechanisms, single-loop closed chains, point paths, coupler ruled surfaces, multiple-mode mechanisms, transition configurations

**MSC 2010**: 53A17, 14J26

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# Contributed talks

## Mathematics Education Relevant, Interesting and Applicable – Are we Ready?

Matija Bašić
Faculty of Science, University of Zagreb, Zagreb, Croatia
e-mail: mbasic@math.hr

Blaženka Divjak
Faculty of Organization and Informatics, University of Zagreb, Varaždin, Croatia
e-mail: bdivjak@foi.hr

Željka Milin Šipuš
Faculty of Science, University of Zagreb, Zagreb, Croatia
e-mail: milin@math.hr

Eva Špalj
XV. gimnazija, Zagreb, Croatia
e-mail: espalj@mioc.hr

The research covered in this presentation has been conducted in the scope of MERIA project financed by Erasmus Plus programme. The main goal of MERIA is to enhance quality and relevance of mathematics education in secondary schools by using inquiry based mathematics teaching (IBMT) and by supporting secondary school teachers? professional development. Besides, project MERIA promotes a positive attitude towards mathematics and shows that mathematics is engaging, important and useful.

In the MERIA project there are eleven partners from four European countries, Croatia, Denmark, the Netherlands and Slovenia, and the project is coordinated by the Faculty of Science, University of Zagreb. Project MERIA is a meeting point for schools and a special category of associated schools in all countries is involved in different activities. These activities include training activities, research about the state of the art of teaching practices and research on the impact of the project on school practices.

At the beginning of the project, in-depth semi-structured interviews have been conducted in 13 schools in all four countries in order to gather data from teachers about the state of the art of schools' teaching practices. The interviews also helped to formulate hypotheses about situation in the schools related to IBMT and to get ideas about how to proceed with trainings and material design. By the research questions significant information about what is important for planning the outputs of the project (training, scenarios, modules, conference etc.) was collected.

There are several hypotheses in this research formulated in a way to reflect expectation and allow testing. One of hypothesis is that teachers are aware of IBME approach and appreciate it, but they do not implement it in classroom practice of mathematics teaching. Another hypothesis is that lack of adequate system support, teaching and learning material, teacher training and overloaded prescribed program are main reasons for not applying IBMT approach in teaching on regular bases. Research questions are aiming at finding good practices of teaching and learning mathematics as well as to identify barriers that hinder implementation of IBMT and other concepts that can motivate students to learn mathematics and to achieve intended learning outcomes.

Report on preliminary findings will be prepared and presented.

**Key words:** mathematics education, inquiry based teaching, secondary schools

**MSC 2010**: 97D10, 97D50

# Geometry and Graphics in the Postgraduate Programmes for Architects and Engineers: an Experience in Progress

Luigi Cocchiarella
Department of Architecture and Urban Studies, Politecnico di Milano, Milan, Italy
e-mail: luigi.cocchiarella@polimi.it

Geometry and Graphics in the postgraduate programmes for Architects nowadays, basically means Digital Geometry and Graphics. Many of these educational programmes are activated in agreement with the Chambers of Architects, Engineers, and Urban Planners, in response to the need for higher digital skills in the professional world, what is also related to the increasing digitalization of the public administration, a field of strong interest also for governments and investors.

From the side of University, postgraduated courses on Digital Geometry and Graphics represent a way to fill the gap appeared in the recent past between education and innovation, especially due to the long lasting resistance carried out by non native digital educators, who were worried about the risk of a loss of traditional didactic approaches and skills. However, trying to judge from a honest perspective, we have to consider that until some years ago, technology in Graphics was not so diffused as it is in the present time, therefore several educational institutions have preferred not to invest on this field, especially those focusing on certain applied sciences and techniques, like the schools of Architecture, at the moment among the less advanced in the digital engagement. The stop was also encouraged by the low technology level of the building construction market around, still based on very traditional procedures, especially in the small private professional contexts. Whatever is our opinion about this, Digital Graphics have for long time developed independently on research and education in Architecture, where mostly software for Mechanical Engineering was initially used, trying to adapt tools and languages to architectural design.

As we know the situation is now changed, and also taking inspiration from the strong recommendation of the *Unesco/UIA Charter for Architectural Education* (Tokyo 2011), indicating as "imperative" to teach the use of computer in all aspects of architectural education, an articulated program for Architects is under discussion at the Politecnico di Milano, concerning all the educational levels, that is, entry, bachelor, master, postgraduate areas.

A couple of years ago we started with the series *Architectural Modelling and BIM* postgraduate courses, a very successful initiative since the beginning. After two years of enthusiastic experiences, as the scientific coordinator of the mentioned courses, as well as a university teacher, I have the clear feeling that the balance between academic approach and market will be the key question in the future. In spite of the wide diffusion of specific tools and approaches, and the consequent request for specific skills and operational procedures from the professional world, academic postgraduate courses for professionals should also keep "critic speculation" as a source, although in a different way from the courses offered in the standard curricula or in the PhD courses, in order to prevent the risk of educating supine users and to remember that they form a relevant part of the university high education, which has also the social responsibility to try to lead the processes, not only to follow them.

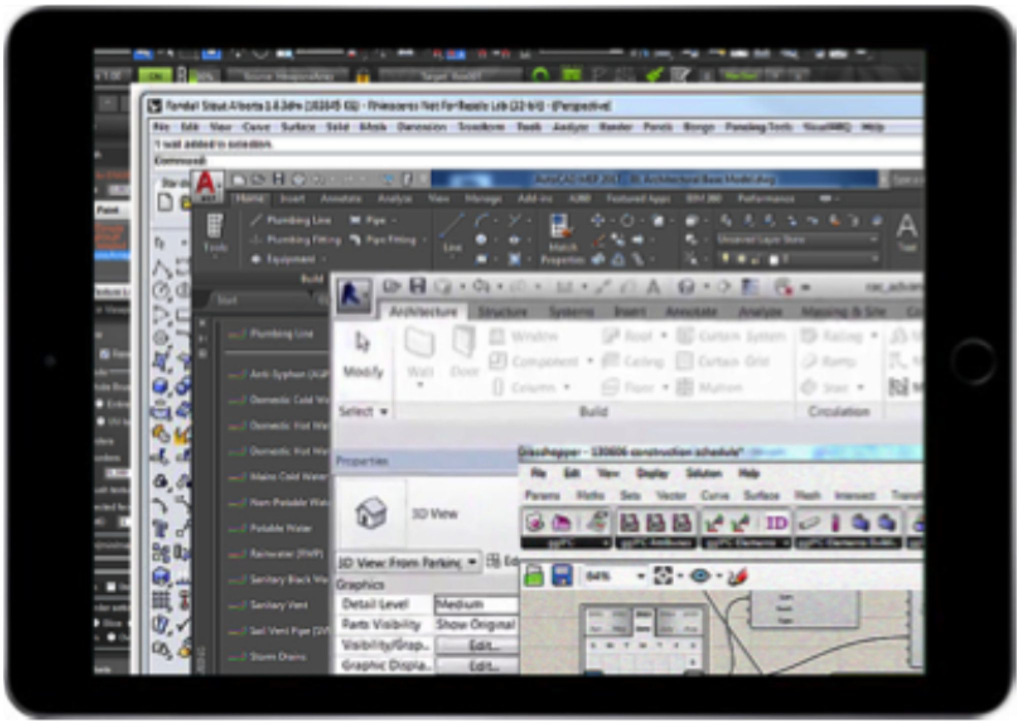

Figure 1: A glance at a crowded set of some of the professional digital sources available

**MSC 2010**: 00A66, 51N05, 01A05, 97U99

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# On Generalization of Cayley Transform in 3D Homogeneous Geometries

Zlatko Erjavec
Faculty of Organization and Informatics, University of Zagreb, Varaždin, Croatia
e-mail: zlatko.erjavec@foi.hr

Cayley transform is the linear fractional transformation $z \mapsto \frac{z-i}{z+i}$ that maps the upper half-plane model of hyperbolic plane isometrically and conformally to the disk model of hyperbolic plane.

In this talk we explain generalizations of Cayley transform in 3D homogeneous spaces $\widetilde{SL(2,\mathbb{R})}$ and $\mathbb{H}^2 \times \mathbb{R}$ both of which are based on the hyperbolic plane. Moreover, we prove that these generalizations are isometries between existing models of corresponding 3D homogeneous geometries.

**Key words:** Cayley transform, $\widetilde{SL(2,\mathbb{R})}$ space, $\mathbb{H}^2 \times \mathbb{R}$ space

**MSC 2010**: 53A40, 53C30

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# A Geometrical Approach to Fully Variable Valve Control

Anton Gfrerrer
Institute of Geometry, Graz University of Technology, Graz, Austria
gfrerrer@tugraz.at

Johann Lang
Institute of Geometry, Graz University of Technology, Graz, Austria
johann.lang@tugraz.at

Variable valve timing (VVT) and variable valve lift (VVL) are used to improve the performance, fuel economy and emissions of a combustion engine. In the last few decades these issues became more and more important. Here we shall address both types VVT and VVL and subsume them in the concept of *variable valve control* (VVC). The first efforts in VVC in a combustion engine were made in the late 1950s and in the 1960s. Not before 1980, though, they were applied in a production car. It was in the late 1980s when the first company introduced some sort of variable valve control on a large scale. Today each automobile manufacturer has his own approach to variable valve control. The outcome is very much of the same kind: Two differently shaped cam lobes are applied by turns, depending on the rev range or on other parameters. Additionally, the whole camshaft can be twisted by a few degrees to allow earlier valve opening (and closing) in the high rev range.

We suggest a new approach to fully variable valve control by means of geometrical and kinematical methods. For a given set of cam lobes we create an appropriate cam surface which is slidable along the camshaft. This cam surface does not only respect the prescribed cam lobes but also accomplishes a continuous set of them.

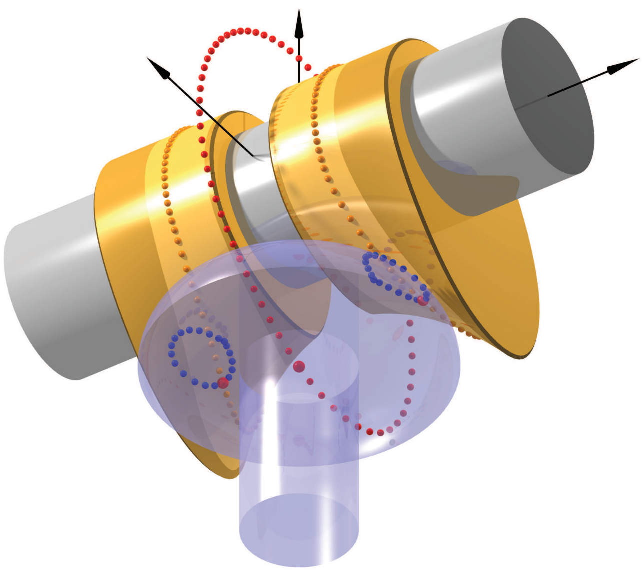

Figure 1: The suggested cam mechanism

# Ellipsoids, Hyperboloids and other Quadrics in Geometry, Arts and Nature

Georg Glaeser
Department of Geometry, University of Applied Arts Vienna, Vienna, Austria
e-mail: georg.glaeser@uni-ak.ac.at

The surfaces of degree two (quadrics) are the simplest algebraic surfaces and may be considered as spatial generalization of conics in the plane.

Being easily understandable, they are – locally seen – prototypes for all surfaces in space. Like conics, they possess an amazing variety of remarkable geometric properties, among them the fact that they carry as many conics as there are points in space. Many of these properties allow useful applications in technology.

Due to their aesthetically pleasing form, surfaces of degree two can readily be evaluated as art objects and, thus, play an important role in design and architecture. We will present examples of such uses, e.g., a geometrically inspired visit of Barcelona, which could be called "the city of quadrics" (Fig. 1).

Figure 1: Barcelona: ellipsoid, hyperboloids at the airport and in the Sagrada Familia

It is of little astonishment that quadrics can also be found in nature, though only in good approximation (as is always the case in nature). The Earth's shape (geoid) comes close to a quadric. Shells of bird eggs, insect eggs, or sea urchins can be well approximated by several mutually touching ellipsoids. "Locally seen", spider nets or spun yarns of ermine moths bear no little resemblance to paraboloids (Fig. 2) and/or hyperboloids, etc.

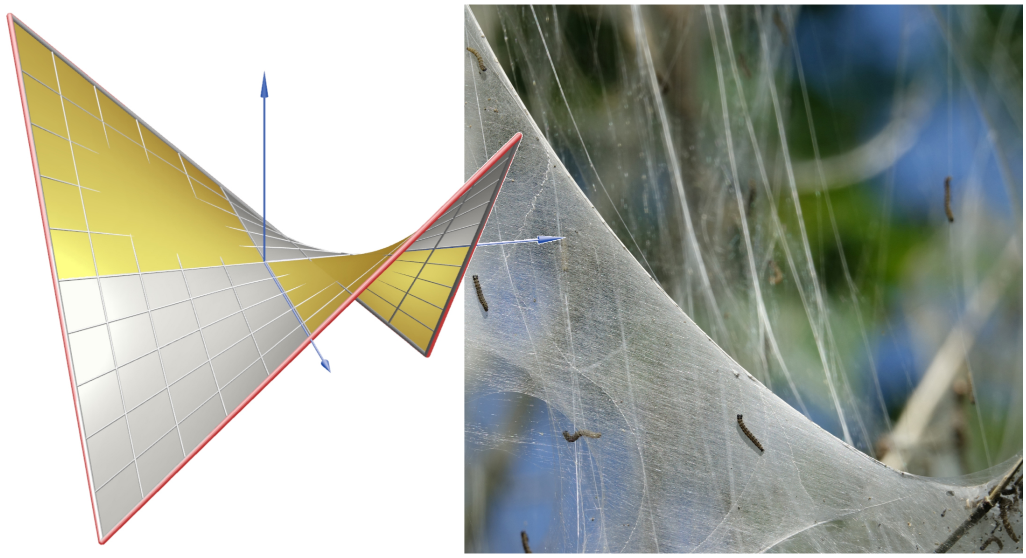

Figure 2: Comparison between minimal surface (gray) and hyperbolic paraboloid (yellow), right: spun yarns of ermine moths come close to minimal surfaces and therefore close to hyperbolic paraboloids.

# Web Textbook for Descriptive Geometry

Sonja Gorjanc
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
e-mail: sgorjanc@grad.hr

We are going to present unreviewed educational material which is a basis for a web textbook primarily designed for the lectures on geometric subjects at the Faculty of Civil Engineering in Zagreb, however, some of its content is also applicable to other faculties in Croatia. The material follows the topical arrangement and it is divided into four chapters: *Plane geometry* (curves and transformations), *Projections* (orthogonal projections, axonometry, elevated projection, perspective), *Surfaces and space curves*, *3D modelling with the program Rhinoceros*. It contains more than 750 graphic files (470 pictures, 80 animations, 61 slideshows, 103 videos and 32 interactive files).

**Key words:** geometric education

**MSC 2010**: 97U20, 97G80

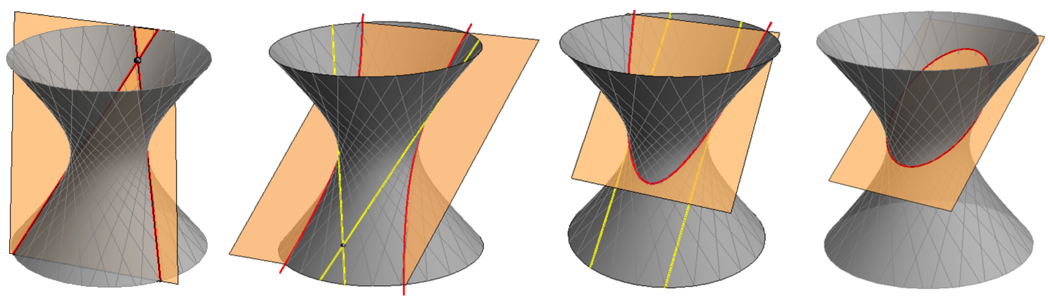

Figure 1: The sections of a hyperboloid of one sheet and plane

# On the Conjugate Diameters of an Ellipsoid. The Construction of Chasles in Practice

Ákos G. Horváth
Department of Geometry, IM Budapest University of Technology and Economics, Budapest, Hungary
e-mail: ghorvath@math.bme.hu

*Dedicated to the memory of the teaching of Descriptive Geometry in BME Faculty of Mechanical Engineering*

The construction of Rytz is known to every mathematician who learnt descriptive geometry. It is less well-known that in the first half of the nineteenth century Chasles (see in [1]) gave a construction for the analogous problem in space. The only reference in English found by the author appears in an old book by Salmon (see in [2]) on the analytic geometry of the three-dimensional space. In the lecture we analyze this construction in the sense of constructability by compass and ruler. The talk is based on a joint manuscript with István Prok.

**Key words:** conjugate diameters, constructability, ellipsoid

## References

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# Nedian Triangle of Ratio $\eta$

Iva Kodrnja
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
e-mail: ikodrnja@grad.hr

Helena Koncul
Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
e-mail: hkoncul@grad.hr

We start with a triangle $\triangle ABC$ and a number $\eta \in \mathbb{R}$. On each of the sides of a triangle (in a counterclockwise order), we choose the point that divides the side in ratio $\eta$ such that

$$\frac{AC_\eta}{AB} = \frac{BA_\eta}{BC} = \frac{CB_\eta}{CA} = \eta$$

and look at the cevians connecting this point and the opposite vertex. These cevians are called nedians with ratio $\eta$. Each pair of the three nedians intersect at a point creating a triangle $\triangle A_1B_1C_1$ called (interior) nedian triangle of ratio $\eta$ (see Fig. 1). Using analytic geometry we can find ratios of perimeters, areas, side-lengths etc of this triangle.

If we vary the parameter $\eta$, we can observe the locus of the vertices of the nedian triangle or its triangle points. We show that this locus lies on the self-isotomic ellipses of the triangle $\triangle ABC$ (see Fig. 2).

Furthermore, for a given triangle $\triangle ABC$ and a fixed number $\eta$ we can repeat the construction of the nedian triangle of ratio $\eta$ on the triangle $\triangle A_1B_1C_1$ and so on. We will analyse properties of these iteration (see Fig. 3).

**Key words:** triange, cevian, nedian, interior nedian triangle, isotomic point

**MSC 2010**: 51M05, 51M20, 51N20

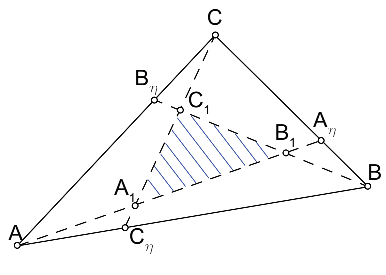


Figure 1: Triangle with nedians of ratio $\eta$ and nedian triangle $\triangle A_1B_1C_1$

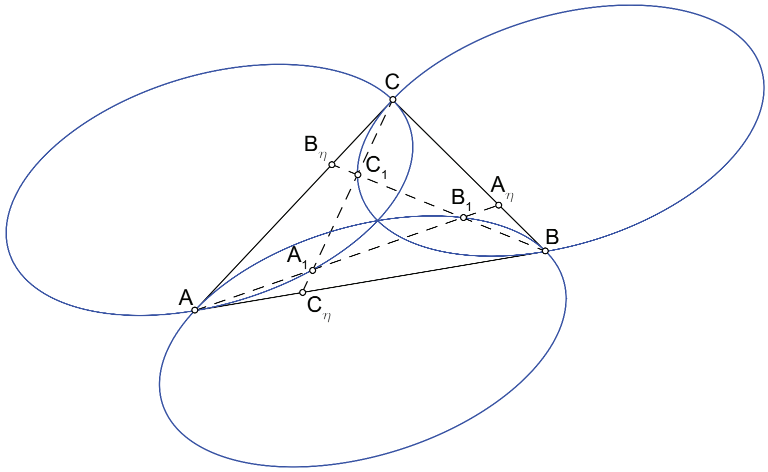


Figure 2: Vertices of $\triangle A_1B_1C_1$ lie on the self-isotomic ellipses of the $\triangle ABC$

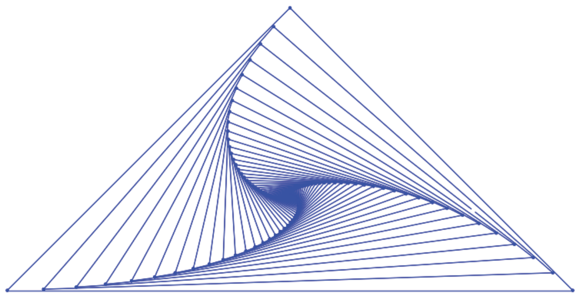

Figure 3: First 100 iterations of nedian triangles of ratio $\eta = 1/16$

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# Digitalization of the Mental Rotation Test

Domen Kušar
Faculty of Architecture, University of Ljubljana, Ljubljana, Slovenia
e-mail: domen.kusar@fa.uni-lj.si

Lidija Pletenac
Faculty of Civil Engineering, University of Rijeka, Rijeka, Croatia
e-mail: lidija.pletenac@uniri.hr

Mateja Volgemut
Faculty of Architecture, University of Ljubljana, Ljubljana, Slovenia
e-mail: mateja.volgemut@fa.uni-lj.si

Mental rotation test is one of the generally recognized tests that are monitoring one of the spatial abilities that cover the field of mental rotation. It involves the ability to create mental images of objects and mentally rotate them. It verifies the ability of mental rotation of the 3D object for the purpose of finding the matching picture of this object.

The Faculty of Architecture has tested freshmen for spatial perception every year since 1999. Its original form was in paper sheet and was divided in three parts: introduction, first part and second part. In the study year 2016/2017 we wanted to update the MRT test with the use of the digital technology. The initiative was given by Lidija Pletenac from the Faculty of Civil Engineering, University of Rijeka, Croatia (below: CE). Domen Kušar and Mateja Volgemut from the Faculty of architecture, University of Ljubljana, Slovenia (below: FA) participated in the work.

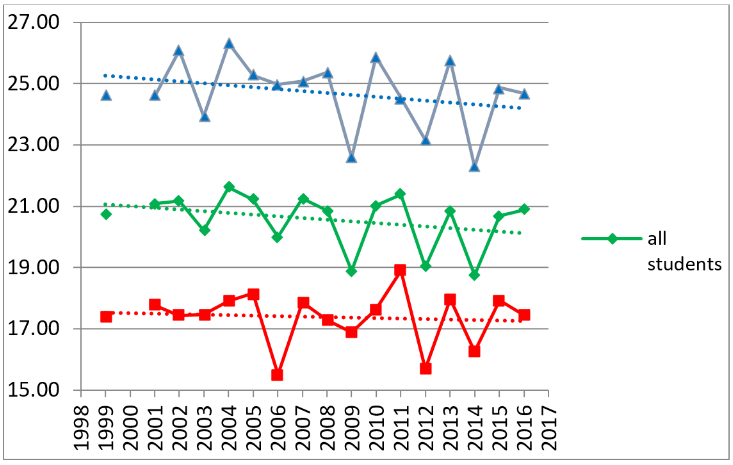
The aim of the collaboration was to establish a form of MRT test that is suitable for the computer program Moodle. The program is a software learning management system widely distributed and used as a teaching tool also on the both faculties. It was necessary to adjust the MRT test due to the specific requirements of the program. On the both mentioned faculties, the new format of the test was performed. During the performance of the test the advantages and disadvantages have been shown. The speed of the task solving and the rapid information of the test results are among them. The Moodle program does not allow the particular method of the evaluation which is used, so the grading is also a disadvantage.

The results also showed different levels of spatial abilities of students. Unfortunately, some groups of students were very small and that disables more realistic comparison. The comparison also showed the improvement in spatial abilities of the students after one semester of study which is greater at the end of the second semester.

Table 1: Tested groups in the study year 2016/2017 and the achieved results. Test was performed in classic form - paper sheets (no colour) and in digitized form (orange colour).

| time | | October 2016 | | | | January 2017 | | | | May 2017 | | May 2017 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| scoring system | | old | | new | | old | | new | | old | | new | |
| GROUP | N | 10 tasks | 20 tasks | 10 tasks | 20 tasks | 10 tasks | 20 tasks | 10 tasks | 20 tasks | 10 tasks | 20 tasks | 10 tasks | 20 tasks |
| FA 1. year | 71 | | | | | 10.93 | 22.87 | 12.44 | 26.24 | | | | |
| FA 1. year | 165 | 10.6 | 20.9 | 12.2 | 24.6 | | | | | | | | |
| FA 1. year | 60 | | | | | | | | | 14.6 | 28.7 | 15.8 | 31.5 |
| CE 1. year | 55 | 7.6 | | 9.8 | | | | | | | | | |
| FA 5. year | 9 | 13.7 | 24.2 | 14.6 | 26.8 | | | | | | | | |
| FA 1. year | 63 | 11.0 | 21.7 | 12.6 | 25.5 | 11.1 | 23.1 | 12.6 | 26.5 | | | | |
| FA 1. year | 29 | 11.3 | 22.7 | 12.7 | 25.8 | 11.6 | 23.7 | 12.7 | 26.6 | 14.9 | 29.2 | 16.0 | 31.9 |

Chart 1: Comparison of the achieved results (FA 1. year) from year 1999 to year 2016.




**Key words:** spatial ability, descriptive geometry, education

**MSC 2010**: 51N05

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# Symmetric Associated Schemes and Finite Geometries

Ivica Martinjak
Faculty of Science, University of Zagreb, Zagreb, Croatia
e-mail: imartinjak@phy.hr

Let $\mathcal{B}$ be a nontrivial biplane of order $k-2$ represented by symmetric canonical incidence matrix with trace equal to $1+\binom{k}{2}$. We prove that $\mathcal{B}$ includes a partially balanced incomplete block design with 3-class association scheme. Consequently, these structures are symmetric, having $2k-6$ points. We also present some representatives of this class of symmetric association schemes. In addition we efficiently construct four biplanes of order 9 - except the one with the smallest automorphism group.

**Key words:** finite geometry, projective plane, association scheme, Bose-Mesner algebra, biplane, automorphism group

**MSC 2010**: 05A17, 11P84

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# On Hyperbolic Crystallography, Cobweb Manifolds

Emil Molnár
Department of Geometry, IM Budapest University of Technology and Economics, Budapest, Hungary
e-mail: emolnar@math.bme.hu

Jenő Szirmai
Department of Geometry, IM Budapest University of Technology and Economics, Budapest, Hungary
e-mail: szirmai@math.bme.hu

The surprising observation of E. Molnár [1] (published first in 1988 in Dubrovnik Proceedings on Differential Geometry) that the football polyhedron {5, 6, 6} can fill the hyperbolic space by a fixed point free group of two generators, seems to get a remarkable application in crystallography as fullerene or $C_{60}$ molecule. This can strengthen the feeling of the authors (and others) that our experimental space can have hyperbolic structure in small size in certain physical circumstances.

Systematic investigations of J. Szirmai [4], [5], e.g. in Nil geometry, show similar phenomena of very dense ball packings and loose ball coverings. As a by-product of our former papers [2], [3] we have found infinite series of hyperbolic groups $\mathbf{Cw}(2z, 2z, 2z)$ acting on polyhedral "cobweb" tilings $\mathbf{Cw}(2z, 2z, 2z)$, $3 \leq z$ odd natural number, so that the orbit spaces will be compact manifolds.

The description of fundamental groups and other properties, moreover visualization of such "finite Worlds" seem to be interesting problems, as well.

**Key words:** fixed point free isometry group of hyperbolic space, infinite series of compact hyperbolic manifolds and possible material structures

**MSC 2010**: 51F15, 52B15, 57S30

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# Hyperbolic Pascal Triangles and Fibonacci Word Fractals

László Németh
Institute of Mathematics, University of Sopron, Sopron, Hungary
e-mail: nemeth.laszlo@uni-sopron.hu

The hyperbolic Pascal triangle $\mathcal{HPT}_{4,q}$ $(q \geq 5)$ is a new mathematical construction [1], which is a geometrical generalization of Pascal's arithmetical triangle. We show that a natural pattern of rows of $\mathcal{HPT}_{4,5}$ is almost the same as the sequence consisting of every second term of the well-known Fibonacci words. Further, we give a generalization of the Fibonacci words using the hyperbolic Pascal triangles. The geometrical properties of a $\mathcal{HPT}_{4,q}$ imply a graph structure between the finite Fibonacci words [2].
Considering more generally the hyperbolic Pascal triangle $\mathcal{HPT}_{p,q}$ $((p-2)(q-2) > 4)$, its each row as a generalized Fibonacci word can be associated with a curve, where the drawing rule is the so-called ?odd-even drawing rule? in L-Systems (see [3]). The limit of these curves of rows as the row approaches infinity gives a Fibonacci word fractal. Figure 1 shows the curve of row 3, where the triangle is $\mathcal{HPT}_{6,9}$.

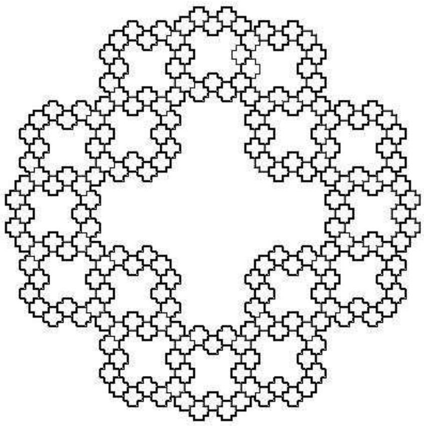

Figure 1: Curve of row 3 in $\mathcal{HPT}_{6,9}$

**Key words:** Hyperbolic Pascal triangle, Fibonacci word, Fibonacci word fractal.

**MSC 2010**: 05B30, 11B39, 28A80.

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# Tetrahedron Centres in a General Metrical Framework

Gennady Notowidigdo
School of Mathematics and Statistics, University of New South Wales, Sydney, Australia
e-mail: gnotowidigdo@gmail.com

In this talk, we present the three main centres of a general tetrahedron (the centroid, circumcentre and the Monge point, which is an analog to the orthocentre of a triangle) over a general metrical framework, as well as key results pertaining to them. Working over an arbitrary symmetric bilinear form allows us to work with a special type of tetrahedron, obtained from a general tetrahedron by way of an affine transformation; furthermore we will be using the framework of Rational trigonometry, developed by Wildberger in 2005, in order to understand the metrical structure associated to the tetrahedron.

While we revisit the familiar concepts of triangle centres, we notice many parallels between them and the centres of the tetrahedron. For example, we will observe that the three main centres all lie on a single line, called the Euler line; furthermore we can derive, for the tetrahedron, an analog of the nine-point centre of a triangle, which we can also study in some depth.

Of note in this talk are some key results regarding the relationship between the circumquadrance (the square of the circumradius) and the other metrical quantities of the tetrahedron, as well as the relationship between the symmetric bilinear form and the existence of orthocentres in a tetrahedron.

# Generalized Conchoids

Boris Odehnal
Department of Geometry, University of Applied Arts Vienna, Vienna, Austria
e-mail: boris.odehnal@uni-ak.ac.at

The well-known construction of conchoids in the Euclidean plane shall be generalized to various geometries. We will give an *intrinsic* construction of conchoids in geometries such as line geometry, sphere geometry, and even in the geometry of Euclidean motions. All these geometries can be modeled within quadrics and crossratios have a geometric meaning. Still the conchoid transform acts on points, but now a point in a model may represent a straight line or a sphere in three-space. It turns out that some conchoid transforms are collineations in the model space. Moreover, these conchoid transforms preserve rational transformations, and thus, the conchoid transform of ruled or canal surface with a rational parametrization is again a ruled surface or canal surface with a rational parametrization. The most simple form of the generalized conchoid transform also preserves the degrees of the objects to be transformed. Nevertheless, we also consider conchoid transformations that alter the degrees of the transformed objects.

**MSC 2010**: 53A05, 51N05, 51Bxx, 53A17, 93B17

# New Examples of Maximal Surfaces in Minkowski Space

Ivana Protrka
Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb, Zagreb, Croatia
e-mail: ivana.protrka@rgn.hr

Ljiljana Primorac Gajčić
Department of Mathematics, Josip Juraj Strossmayer University of Osijek, Osijek, Croatia
e-mail: lprimora@mathos.hr

Željka Milin Šipuš
Faculty of Science, University of Zagreb, Zagreb, Croatia
e-mail: milin@math.hr

A surface in the 3-dimensional Minkowski space $\mathbb{R}^3_1 = (\mathbb{R}^3,\, dx^2 + dy^2 - dz^2)$ is called a spacelike surface if its induced metric is positive definite. A spacelike surface with vanishing mean curvature is called a maximal surface. There are many ways to obtain maximal surfaces. One of them is by the so-called Björling formula

$$\chi(z) = \mathrm{Re}\Big(\beta(z) + i\int_{s_0}^{z} V(w) \times \beta'(w)dw\Big).$$

The Björling formula gives real parametrization $\chi(z)$ of a maximal surface using the default real analitic curve $\beta\colon I \to \mathbb{R}^3_1$ and a prescribed unit vector field $V$ along $\beta$ ([1], [2]). In this presentation we give new examples of maximal surfaces (see Figure 1) based on epicycloids and hypocycloids in the 3-dimensional Minkowski space, by means of the Björling formula and the Weierstrass-Enneper representation of a surface. The Euclidean counterparts of these surfaces have been investigated in [3].

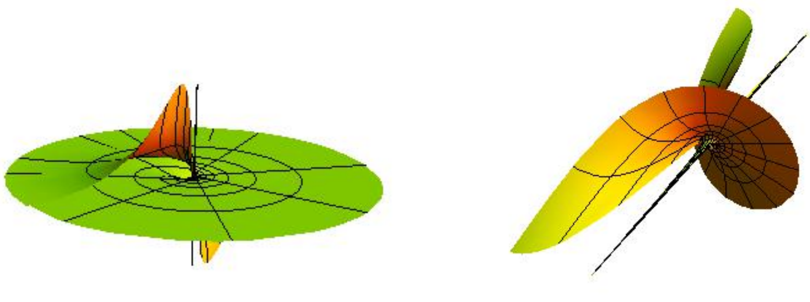

Figure 1: Maximal surfaces on a geodesic hypocycloid

**Key words:** Minkowski 3-space, maximal surface, Björling formula

**MSC 2010**: 53A35, 53B30

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# Volumetry in Gaudi's Works at Park Güell

Madalena Ribeiro Grimaldi
Federal University of Rio de Janeiro, Rio de Janeiro, Brazil
e-mail: mgrimaldi@eba.ufrj.br

Glaucia Augusto Fonseca
Federal University of Rio de Janeiro, Rio de Janeiro, Brazil
Pontifical Catholic University, Rio de Janeiro, Brazil
School of Industrial Design of Rio de Janeiro, Rio de Janeiro, Brazil
University Estácio de Sá, Rio de Janeiro, Brazil
e-mail: glauciaaugsto@gmail.com

This article highlights existing relations between Geometry and the architectonic elements in Antonio Gaudi's works at Park Güell in Barcelona City. Having been designed to be a condominium inspired in the Garden City ideal, and displaying organic shapes, this park exhibits volumetry founded on structural solutions delimited by geometric concepts. On the basis of some examples found therewith, the authors describe an interdisciplinary didactic proposal carried out in the year 2016, in the course of "Interior Design" at "Escola de Belas Artes da Universidade Federal do Rio de Janeiro", and in the course of "Architecture and Urbanism" at "Escola Superior de Desenho Industrial da Universidade Estadual do Rio de Janeiro". By means of this alternative work, the students were able to observe in the structures the generation of surfaces composed of multiple lines which had been studied previously. The tasks were performed through designing sketches, creating physical models and digital modeling of three sectors of Park Güell. This didactic approach aimed at showing that exercising the geometric analysis on the shapes designed by Gaudi, as well as making volumetric copies of his works, would serve to stimulate the creative process in the various stages necessary for conceiving projects of this kind. Furthermore, the exercises would enhance the spatial perception of students.

**Key words:** Park Güel, geometry, volumetry, modeling exercises

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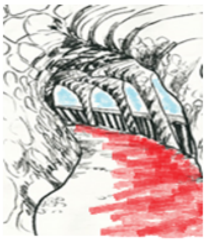
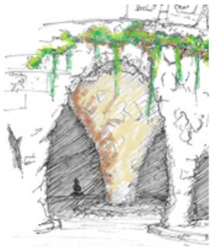
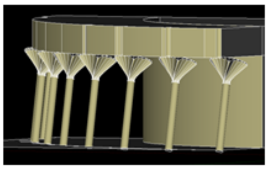
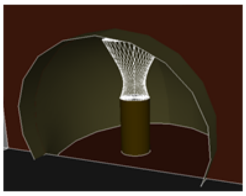
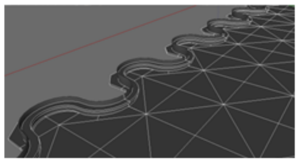
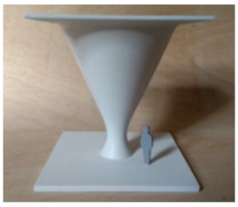
| **Works made by students** | | | |
|---|---|---|---|
| | HELICOPTER | HYPERBOLOID | FLOW CURVES |
| Sketches |  |  |  |
| Digital Modeling |  |  |  |
| Physical models |  |  |  |

# Reflection in Quadrics and Ivory's Theorem

Hellmuth Stachel
Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria
e-mail: stachel@dmg.tuwien.ac.at

In the Euclidean 3-space the reflection in a quadric $Q$ is connected with the range of quadrics being confocal with $Q$, in particular with the focal conics of $Q$. Some properties of this reflection will be presented together with examples of closed billiards in an ellipsoid.

Recently, Izmestiev and Tabachnikov gave an intuitive proof of Ivory's theorem in the plane via reflections in confocal conics. In the lecture 3D versions of this approach will be discussed.

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# Workshop "Parametric Modelling and Digital Fabrication"

Milena Stavrić
Institute of Architecture and Media, Graz University of Technology, Graz, Austria
e-mail: mstavric@tugraz.at

Albert Wiltsche
Institute of Architecture and Media, Graz University of Technology, Graz, Austria
e-mail: wiltsche@tugraz.at

Mirela Katić Žlepalo
Department of Civil Engineering, University of Applied Sciences, Zagreb, Croatia
e-mail: mkatic@tvz.hr

The workshop "Parametric Modelling and Digital Fabrication" was held at the Department of Civil Engineering at the University of Applied Sciences in Zagreb from February 18th to 21st, 2017. The aim of this four-day workshop was to introduce students to parametric modelling and to CNC fabrication process using a laser cutter. The international workshop was intended for students of civil engineering at Zagreb University of Applied Sciences and for guest students of architecture from the Faculty of Technical Sciences, University of Novi Sad and Faculty of Architecture, Civil Engineering and Geodesy, University of Banja Luka.

The topic of the workshop was mid-rise tower, which allowed both students of civil engineering and architecture to find their own motivation to build their own towers beginning with a virtual parametric design and ending with an analogue scale model.

In this talk, we will present our pedagogical approach, basic ideas for modelling parametrical objects strongly based on geometrical rules and the results of the digital fabrication.

**MSC 2010**: 97G40, 97G80, 97Q60

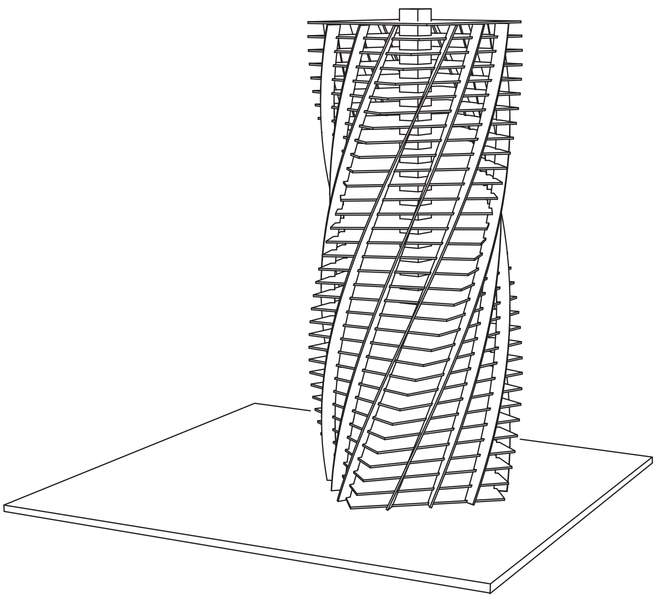

Figure 1: Rhino model

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# Local Shaping of B-spline Surfaces by Building in a Given Pattern

Márta Szilvási-Nagy
Department of Geometry, Budapest University of Technology and Economics, Budapest, Hungary
e-mail: szilvasi@math.bme.hu

In this presentation we show a technique for modifying the shape of a base surface locally such that a part of it approximates another given surface (pattern surface). Both of the surfaces are B-spline surfaces of degree $3 \times 3$ described in matrix form with non-uniform knot vectors (see [1]). Some control points of the base surface are the variables in the approximation, and are computed by minimizing an objective function including the sum of squared distances between corresponding points of the base surface and the pattern surface. This is the well-known least squares method frequently applied in the solutions of interpolation and approximation problems. Then we apply additional fairing conditions in order to minimize the approximation error and to get the most satisfactory shape of the resulting surface.

**Key words:** B-spline surface, matrix representation, approximation

**MSC 2010**: 65D17, 65D05, 65D07, 68U05, 68U07

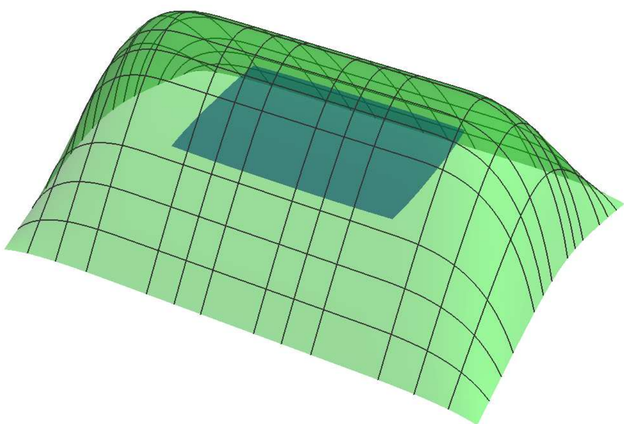

Figure 1: The base surface and the given pattern surface.

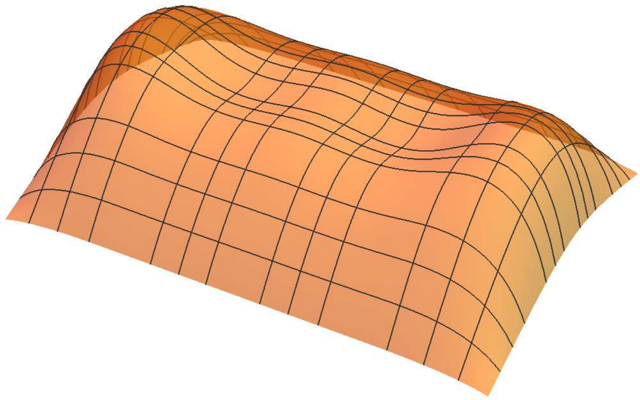

Figure 2: The modified surface approximating the pattern surface, while the boundary curves are unchanged.

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# Translation Triangles, Tetrahedra and Bisector Surfaces in Sol Geometry

Jenő Szirmai
Department of Geometry, IM Budapest University of Technology and Economics, Budapest, Hungary
e-mail: szirmai@math.bme.hu

In this talk we study the **Sol** geometry that is one of the eight homogeneous Thurston 3-geometries (see [1], [2], [3], [4]).

We analyse the interior angle sums of translation triangles in **Sol** geometry and prove that it can be larger or equal than $\pi$ (see [5]). We determine the equation of the translation-like bisector surface of any two points. We prove, that the isosceles property of a translation triangle is not equivalent to two angles of the triangle being equal and that the triangle inequalities do not remain valid for translation triangles in general. Moreover, we develop a method to determine the centre and the radius of the circumscribed translation sphere of a given *translation tetrahedron* (see [6]).

In our work we will use for computations and visualizations the projective model of **Sol** described by E. Molnár in [2].

**Key words:** Thurston geometries, **Sol** geometry, translation-like bisector surface of two points, circumscribed sphere of **Sol** tetrahedron, Dirichlet-Voronoi cell

**MSC 2010**: 53A20, 53A35, 52C35, 53B20

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# On Volume Formulas for the Intersection of a Simplex and a Half-Space

István Talata
Ybl Faculty of Szent István University, Budapest, Hungary
e-mail: talata.istvan@ybl.szie.hu

We present a new, geometric, elementary and short proof of the volume formula

$$vol(S_n \cap H^+) = \sum_{a_i>0} \left( \prod_{j \neq i} \frac{a_i}{a_i - a_j} \right) vol(S_n)$$

for the intersection of an arbitrary simplex $S_n$ and a half-space $H^+$ in the $n$-dimensional Euclidean space $\mathbb{E}^n$, $n \geq 1$, where $H^+$ consists of those points $x \in \mathbb{E}^n$ whose barycentic coordinates $(x_1, x_2, \ldots x_n, x_{n+1})$ with respect to the vertices $(v_1, v_2, \ldots v_n, v_{n+1})$ of simplex $S_n$ fulfill $\sum_{i=1}^{n+1} a_i x_i \geq 0$ where $a_i$'s are pairwise distinct. For earlier proofs of this formula, see [1, 2, 3].

We also present a new recursive formula to compute $vol(S_n \cap H^+)/vol(S_n)$. We define the quantities $f(i,j)$ for $1 \leq i \leq j \leq n+1$ in a way that $f(1, n+1) = vol(S_n \cap H^+)/vol(S_n)$. We show that the recursive formula

$$f(i,j) = \frac{a_j}{a_j - a_i} f(i+1, j) + \frac{a_i}{a_i - a_j} f(i, j-1)$$

holds when $i < j$ and $a_i$'s form a strictly increasing sequence, while $f(i,j) = 1$ may be assumed if $a_i \geq 0$ and $f(i,j) = 0$ may be assumed if $a_j < 0$. Computing $vol(S_n \cap H^+)/vol(S_n)$ by this recurrence relation turns out to be computationally more stable than computing by the previous volume formula, and it has similar computational complexity.

**Key words:** volume formula, section of simplex, recursive formula

**MSC 2010**: 52A20, 52A38, 51N20

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# Poles of Inaccessibility in Croatia

Dražen Tutić
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: dtutic@geof.hr

Matjaž Štanfel
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: mstanfel@geof.hr

Ana Kuveždić Divjak
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: akuvezdic@geof.hr

Martina Triplat Horvat
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: mthorvat@geof.hr

A pole of inaccessibility is a location that is most challenging to reach. In geography, it is usually the most distant location from the coastline, either on the sea or on the land. Poles of inaccessibility were usually determined for oceans and continents, but they can also be established for smaller regions, like islands or administrative regions (e.g. Croatia). Numerous administrative units want and already have determined centre, usually called a geographic centre which is determined as geometric centre (centroid) or centre of gravity. Geographic centre of Croatia falls into Bosnia and Herzegovina, due to its elongated and bent shape. Pole of inaccessibility always falls inside the region and makes it an alternative to centroid, not only to define the centre of the region but also to be used in GIS as the point that represents the polygon.

In this paper, using existing methods, we determined and compared poles of inaccessibility in Croatia. Some methods give solution in plane and some on Earth's sphere. Since Croatia is relatively small region, the Earth's curvature does not have a substantial influence. Obviously, the solution depends on the definition of the region and since Croatia has well-indented coastline, we can consider both the mainland only or the whole country including islands and the sea. In Croatia's mainland, the most distant point, with the distance of 61.5 km from administrative border, falls in village Mostari, located 10 km north-east of Ivanić Grad. For state administrative border (including islands and the sea), the most distant point with the distance of 73 km is located in the sea 6 km south-east of village Lopar on the island Rab (Fig. 1).

**Key words:** pole of inaccessibility, Croatia

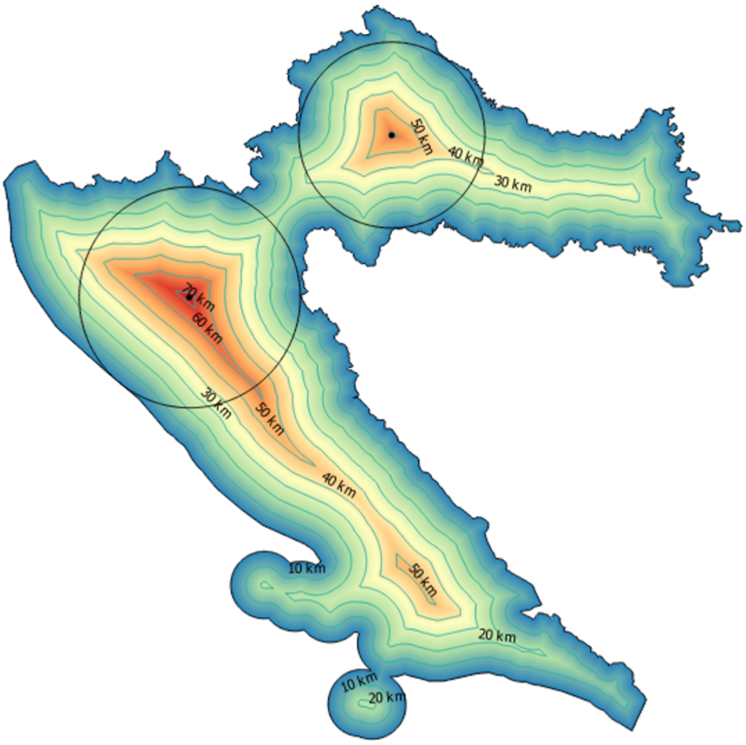


Figure 1: Poles of inaccessibility in Croatia

# Minkowski Product of Free-Form Curves

Daniela Velichová
Slovak University of Technology, Bratislava, Slovakia
e-mail: daniela.velichova@stuba.sk

This paper presents basic information about Minkowski product and combinations of free-form curve segments. Some of intrinsic geometric properties of surface patches generated by means of this method are presented, with illustrations of particular resulting geometric forms. Presented generating principles for modelling curve segments and surface patches are discussed as tools for re-modelling free-form curve segments in $\mathbf{E}^3$ represented parametrically by uniform vector maps on the unit real interval $I = \langle 0, 1\rangle \subset \mathbf{R}$.
Let us restrict considerations specifically to Bézier curve segments in $\mathbf{E}^3$ represented parametrically by vector maps on the unit real interval $I = \langle 0, 1\rangle \subset \mathbf{R}$

$$\mathbf{K}: \mathbf{k}(u) = \sum_{i=1}^{n} A_i Be_{in}(u) = \left(\sum_{i=1}^{n} xa_i Be_{in}(u), \sum_{i=1}^{n} ya_i Be_{in}(u), \sum_{i=1}^{n} za_i Be_{in}(u)\right),$$

$$\mathbf{L}: \mathbf{l}(v) = \sum_{j=1}^{n} B_j Be_{jn}(v) = \left(\sum_{j=1}^{n} xb_j Be_{jn}(v), \sum_{j=1}^{n} yb_j Be_{jn}(v), \sum_{j=1}^{n} zb_j Be_{jn}(v)\right),$$

where $A_i = (xa_i, ya_i, za_i), B_j = (xb_j, yb_j, zb_j)$ are vertices of the curves basic determining polygons and $Be_{in}, Be_{jn}$ are Bernstein interpolation polynomials of degree $n$.

Minkowski product of curves $\mathbf{K}, \mathbf{L}$ is a Bézier patch of degree $2n$ defined on unit square $I^2 \subset \mathbf{R}^2$ as

$$\mathbf{K} \otimes \mathbf{L}: \mathbf{p}(u, v) = \mathbf{k}(u) \otimes \mathbf{l}(u) = \sum_{i=1}^{n} \sum_{j=1}^{n} (A_i \otimes B_j) Be_{in}(u) Be_{jn}(v).$$

For $n = 3$ the resulting bi-cubic Bézier patch is determined by net of points given in the map

$$\begin{pmatrix} A_0 \otimes B_0, A_0 \otimes B_1, A_0 \otimes B_2, A_0 \otimes B_3 \\ A_1 \otimes B_0, A_1 \otimes B_1, A_1 \otimes B_2, A_1 \otimes B_3 \\ A_2 \otimes B_0, A_2 \otimes B_1, A_2 \otimes B_2, A_2 \otimes B_3 \\ A_3 \otimes B_0, A_3 \otimes B_1, A_3 \otimes B_2, A_3 \otimes B_3 \end{pmatrix}.$$

Partial Minkowski product of curve segments $\mathbf{K}, \mathbf{L}$ equally parameterized for $u = v = t$ is a Bézier curve segment of degree $2n$ determined by parametric representation

$$\mathbf{r}(t) = \mathbf{k}(t) \otimes \mathbf{l}(t) = \sum_{i=1}^{n} (A_i \otimes B_i) Be_{in}^2(t)$$

that is located on the bicubic Bézier patch defined by Minkowski product of curve segments $\mathbf{K}, \mathbf{L}$ parameterized by different parameters $u, v \in I$.

Map of this sextic is of the form

$$\begin{pmatrix} A_0 \otimes B_0 \\ A_0 \otimes B_1 + A_1 \otimes B_0 \\ A_0 \otimes B_2 + 3(A_2 \otimes B_2) + A_2 \otimes B_0 \\ A_0 \otimes B_3 + 9(A_1 \otimes B_2) + 9(A_2 \otimes B_1) + A_3 \otimes B_0 \\ A_1 \otimes B_3 + 3(A_2 \otimes B_2 \\ A_3 \otimes B_3 \end{pmatrix}^T .$$

Examples of Minkowski product and partial product of two Bézier cubical segments are shown in Figure 1.

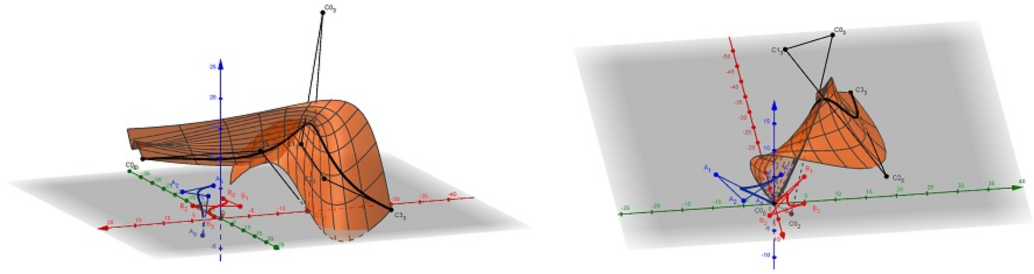

Figure 1: Minkowski product and partial product of two Bézier curve segments

**Key words:** Minkowski product of point sets, free-form curves

**MSC 2010**: 65D17, 51H30, 68U07

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# Non-standard Aspects of Fibonacci-Series

Gunter Weiss
Institute of Discrete Mathematics and Geometry, Vienna University of Technology, Vienna, Austria
Department of Mathematics, Dresden University of Technology, Dresden, Germany
e-mail: weissgunter@hotmail.com

Fibonacci-series and the limes of the quotient of adjacent Fibonacci-numbers, namely the Golden Mean, belong to general knowledge of roughly said anybody, not only of mathematicians and geometers. There were several attempts to generalize these fundamental concepts, which also found applications in art and architecture, as e.g. number series and quadratic equations leading to the so-called "Metallic means" of V. de Spinadel or the cubic "plastic number" of van der Laan resp. the "cubi ratio" of L. Rosenbusch. The mentioned generalisations consider series of integers or real numbers. "Non-standard aspects" mean now generalisations with respect to a given number field or ring as well as visualisations of the resulting geometric objects.

# Extending Rational Trigonometry into Higher Dimensions

N J Wildberger
School of Mathematics and Statistics, University of New South Wales, Sydney, Australia
e-mail: n.wildberger@unsw.edu.au

Rational trigonometry allows us an algebraic reformulation of Euclidean geometry, based on the rational notions of quadrance and spread, to replace the more familiar distance and angle. In fact this form of trigonometry actually goes back to the ancient Babylonians! The usual laws of affine planar trigonometry get replaced by purely algebraic relations, speeding up computations, extending to more general quadratic forms, working over arbitrary fields (not of characteristic two), and capable of extensions to the projective situation, covering both hyperbolic and spherical geometries.

But how can we extend affine trigonometry to higher dimensions? Previous work of the author has shown that in three dimensions the crucial new quantity is the solid spread, which replaces the notion of solid angle, and allows a reformulation of the trigonometry of the tetrahedron, which is the object of joint study now with Gennady Notowidigdo.

In this talk we want to go further and outline an ambitious new program to tackle the framework for higher dimensional trigonometry. We will need to greatly expand our thinking, and many new problems and questions arise.
There is much to be done!

# Posters

## Augmented Reality Sandbox in the Classroom

Matija Balaško
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: mbalasko@geof.hr

Dražen Tutić
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: dtutic@geof.hr

Ana Kuveždić Divjak
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: akuvezdic@geof.hr

Augmented reality (AR) sandbox is an interactive educational tool that can be used to explore relief models and the way to represent relief on maps by contours and/or hypsometric tints. Since it can be realised using affordable hardware (beamer, Kinect and personal computer) and a free software, it makes the AR sandbox a convenient educational tool for schools and faculties. Exploring, playing and learning relief and its representation on maps with AR sandbox is a great way for pupils, students and everyone else, e.g. hikers or orientation runners, to be better skilled in interpretation of relief from maps. Tasks given in a form of contour map which combine different landforms (ridges, valleys, saddles, slopes, pits etc.) when correctly formed in AR sandbox give immediate experience of its 3D shape.

Faculty of Geodesy of the University of Zagreb has made an AR sandbox and created educational tasks for different educational stages. On the study of Geodesy and Geoinformatics there are courses in which AR sandbox can be used, e.g. Topography, Cartography, Spatial Orientation and Perception of the Environment. It will be also available when pupils with their geography teachers or other interested members of public come to visit. It can also serve as entertainment or promotion on various events.

**Key words:** education, sandbox, augmented reality, topography, maps

# Leopold Sorta
# Students' Works at the Beginning of the 20th Century

Jelena Beban-Brkić
Faculty of Geodesy, University of Zagreb, Zagreb, Croatia
e-mail: jbeban@geof.hr

Nikoleta Sudeta
Faculty of Architecture, University of Zagreb, Zagreb, Croatia
e-mail: nikoleta.sudeta@arhitekt.hr

Leopold Sorta (Sušak, current-days a part of the City of Rijeka, 1881 – Zagreb, 1956), Croatian engineer of the Naval Architecture, studied the Mechanical Engineering in Vienna and Munich, then the Naval Architecture at the High Technical School in Charlottenburg near Berlin, which he graduated from, 1914.

Afterwards, he got a job in the shipyard Ganz-Danubius ("3. Maj" nowadays), precisely in it's Construction office. His devoted engagement in designing guard boats, torpedo boat destroyers, submarines and the battleship Szent Istvan, was well approved by written documents.

Leopold Sorta made exhibited drawings in the course of his studentship at the High Technical School Mechanical engineering Department. The drawings are preserved at the Faculty of Architecture and at the Faculty of Geodesy, University of Zagreb.

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Figure 1: Leopold Sorta (Sušak, 1881 – Zagreb, 1956)
From: http://www.lokalpatrioti-rijeka.com/forum/viewtopic.php?p=69154

# List of participants

1. Matija Balaško
Faculty of Geodesy, University of Zagreb
*mbalasko@geof.hr*
2. Jelena Beban-Brkić
Faculty of Geodesy, University of Zagreb
*jbeban@geof.hr*
3. Luigi Cocchiarella
Department Architecture and Urban Studies, Politecnico di Milano
*luigi.cocchiarella@polimi.it*
4. Géza Csima
Institute of Mathematics, Budapest University of Technology and Economics
*csgeza@math.bme.hu*
5. Blaženka Divjak
Faculty of Organization and Informatics, University of Zagreb
*blazenka.divjak@foi.hr*
6. Zlatko Erjavec
Faculty of Organization and Informatics, University of Zagreb
*zlatko.erjavec@foi.hr*
7. Anton Gfrerrer
Institute of Geometry, Graz University of Technology
*gfrerrer@tugraz.at*
8. Georg Glaeser
Department of Geometry, University of Applied Arts Vienna
*gg@uni-ak.ac.at*
9. Sonja Gorjanc
Faculty of Civil Engineering, University of Zagreb
*sgorjanc@grad.hr*
10. Damir Horvat
Faculty of Organization and Informatics, University of Zagreb
*damir.horvat1@foi.hr*
11. Ákos G. Horváth
BME Faculty of Natural Sciences, Budapest University of Technology and Economics
*ghorvath@math.bme.hu*

12. Ema Jurkin
Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb
*ema.jurkin@rgn.hr*

13. Mirela Katić Žlepalo
Department of Civil Engineering, University of Applied Sciences, Zagreb
*mirela.katic-zlepalo@tvz.hr*

14. Kristijan Kilassa Kvaternik
Faculty of Electrical Engineering and Computing, University of Zagreb
*kristijan.kilassakvaternik@fer.hr*

15. Iva Kodrnja
Faculty of Civil Engineering, University of Zagreb
*ikodrnja@grad.hr*

16. Helena Koncul
Faculty of Civil Engineering, University of Zagreb
*hkoncul@grad.hr*

17. Nikolina Kovačević
Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb
*nkovacev@rgn.hr*

18. Zoltán Kovács
The Private University College of Education of the Diocese of Linz
*zoltan@geogebra.org*

19. Domen Kušar
Faculty of Architecture, University of Ljubljana
*domen.kusar@fa.uni-lj.si*

20. Ivica Martinjak
Faculty of Science, University of Zagreb
*imartinjak@phy.hr*

21. Sybille Mick
Institute of Geometry, Graz University of Technology
*mick@tugraz.at*

22. Željka Milin Šipuš
Faculty of Science, University of Zagreb
*milin@math.hr*

23. Emil Molnár
Department of Geometry, Budapest University of Technology and Economics
*emolnar@math.bme.hu*

24. László Németh
Institute of Mathematics, University of West Hungary
*lnemeth@emk.nyme.hu*

25. Gennady Notowidigdo
School of Mathematics and Statistics, University of New South Wales
*gnotowidigdo@gmail.com*

26. Boris Odehnal
Department of Geometry, University of Applied Arts Vienna
*boris.odehnal@uni-ak.ac.at*

27. Martin Pfurner
Unit for Geometry and CAD, University of Innsbruck
*martin.pfurner@uibk.ac.at*

28. Lidija Pletenac
Faculty of Civil Engineering, University of Rijeka
*lidija.pletenac@uniri.hr*

29. Ivana Protrka
Faculty of Mining, Geology and Petroleum Engineering, University of Zagreb
*ivana.protrka@rgn.hr*

30. Madalena Ribeiro Grimaldi
Federal University of Rio de Janeiro
*mgrimaldi@eba.ufrj.br*

31. Hellmuth Stachel
Institute of Discrete Mathematics and Geometry, Vienna University of Technology
*stachel@dmg.tuwien.ac.at*

32. Milena Stavrić
Institute of Architecture and Media, Graz University of Technology
*mstavric@tugraz.at*

33. Márta Szilvási-Nagy
Deptment of Geometry, Budapest University of Technology and Economics
*szilvasi@math.bme.hu*

34. Jenö Szirmai
Deptment of Geometry, Budapest University of Technology and Economics
*szirmai@math.bme.hu*

35. Matjaž Štanfel
Faculty of Geodesy, University of Zagreb
*mstanfel@geof.hr*

36. István Talata
Ybl Faculty of Szent István University
*talata.istvan@ybl.szie.hu*

37. Dražen Tutić
Faculty of Geodesy, University of Zagreb
*dtutic@geof.hr*

38. Daniela Velichová
Department of Mathematics, Slovak University of Technology, Bratislava
*daniela.velichova@stuba.sk*

39. Mateja Volgemut
Faculty of Architecture, University of Ljubljana
*Mateja.volgemut@fa.uni-lj.si*

40. Gunter Weiss
Dresden University of Technology, Vienna University of Technology
*weissgunter@hotmail.com*

41. Norman John Wildberger
University of New South Wales, Sydney
*n.wildberger@unsw.edu.au*

42. Albert Wiltsche
Institute of Architecture and Media, Graz University of Technology
*wiltsche@tugraz.at*