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Visualization of Curves and Spheres in *Sol* Geometry

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ABSTRACT

The paper makes an attempt to visualize one of the homogeneous geometries, the *Sol* geometry, by illustrating first the geodesic curves and spheres then the so-called translation curves and spheres. We've collected their basic properties, too.

Key words: visualization of Thurston's geometries

MSC 2000: 53B20, 53C30

Vizualizacije krivulja i ploha u *Sol* geometriji

SAŽETAK

Ovaj članak je pokušaj vizualizacije jedne od homogenih geometrija, *Sol* geometrije. Prvo se ilustriraju geodetske krivulje i sfere, a zatim i tzv. translatirajuće krivulje i sfere. Također su navedena njihova osnovna svojstva.

Ključne riječi: vizualizacija Thurstonovih geometrija

"This (the *Sol* geometry) *is the real weird. Unlike the previous geometries, solve geometry isn't even rotationally symmetric. I don't know any good intrinsic way to understand it."* (J. R. WEEKS) [7]

1 Introduction

In [5] W. P. Thurston formulated a geometrization conjecture for three-manifolds which states that every compact orientable three-manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the 8 maximal simply connected homogeneous Riemannian 3-geometries E^3, H^3, S^3, $S^2\times R$, $H^2\times R$, $\widetilde{SL(2,R)}$, Nil and Sol. Obviously, the Poincaré conjecture (a compact three-manifold with trivial fundamental group is necessarily homeomorphic to the 3-sphere) is a special case of the Thurston conjecture. In the past thirty years, many mathematicians have contributed to the understanding of this problem, maybe the most important attempts are due to R. Hamilton. In 2006 a scoop went round the world claiming that a Russian mathematician, G. I. Perelman could give a complete proof of the Thurston conjecture and so the Poincare conjecture, too. Followed by the complex and knotty proof (using modern differential geometry of Ricci flows) the interest has turned to homogeneous spaces. This paper tries to help in understanding one of the above geometries, the *Sol*.

Let (M,g) be a Riemannian manifold. If for any $x,y\in M$ there does exist an isometry $\Phi: M\to M$ such that $y=\Phi(x)$, then the Riemannian manifold is called *homogeneous*.

The visualization of the three possible two-dimensional homogeneous Riemann geometries E^2, H^2, S^2 is familiar to anyone, but in higher dimensions we face a lot of open questions. Even in three dimensions, where first time anisotropic cases also appear we have difficulties in the imagination. No doubt, the standard models work for E^3, H^3, S^3, moreover real-time interactive graphics algorithms have been developed by J. R. WEEKS that can be extended even more for the product spaces $S^2\times R$, $H^2\times R$ [8]. The remaining three Thurston's homogeneous 3-dimensional geometries $\widetilde{SL(2,R)}$, Nil and Sol, however are difficult to handle. From these the twisted spaces $\widetilde{SL(2,R)}$ and Nil need multiple imaging and there are just a few results about them, whilest the *Sol* (mentioned also as *solv* in the literature) is the most unusual as our motto above indicates, as well (for more information consult [5], [6], [4]). We note that in the paper [2] Emil MOLNÁR elaborated the projective interpretations of all the eight geometries, we only cite his model for *Sol*.

Sol geometry can be obtained by giving a group structure T to be a semi-direct product $R\ltimes R^2$ as follows:

$$\begin{pmatrix}1 & a & b & c\end{pmatrix}\begin{pmatrix}1 & x & y & z\\ 0 & e^{-z} & 0 & 0\\ 0 & 0 & e^{z} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}=$$
$$\begin{pmatrix}1 & x+ae^{-z} & y+be^{z} & z+c\end{pmatrix}$$

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is the right action by a translation (x,y,z) on an affine point (a,b,c) yielding also a point of *Sol* expressed in homogeneous (projective) coordinates after choosing a fixed origin $O(1,0,0,0)$.

Then an invariant metric on $Sol(O,T)$ is given by

$$(ds)^2 = e^{2z}(dx)^2 + e^{-2z}(dy)^2 + (dz)^2,$$

as infinitesimal arc length square, now in any point $(1,x,y,z)$ [5], [2].

2 Geodesics and their representation

In the following we briefly recall from [1] the standard procedure yielding the geodesics of *Sol*.

Consider first the fundamental (metric) tensor from the above mentioned equation

$$(g_{ij}) = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The well-known equation of geodesics

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = 0$$

containing the Christoffel symbols of second kind: $\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l}\right) g^{lk}$ turns [1] to

$$\begin{aligned} &\ddot{x} + 2\dot{x}\dot{z} = 0 \\ &\ddot{y} - 2\dot{y}\dot{z} = 0 \\ &\ddot{z} - e^{2z}(\dot{x})^2 + e^{-2z}(\dot{y})^2 = 0. \end{aligned}$$

Solving this differential equation system as a Cauchy problem

$$\begin{matrix} x(0)=0 & & \dot{x}(0)=u \\ y(0)=0 & \text{and} & \dot{y}(0)=v \\ z(0)=0 & & \dot{z}(0)=w \end{matrix}$$

$$u^2+v^2+w^2=1$$

we could arrange the following table that contains our results:

(1)	$u \neq 0$ $v \neq 0$ $0 < \lvert w\rvert = \sqrt{1-u^2-v^2} < 1$	$x(t) = u\int_0^t e^{-2z(\tau)}d\tau$ $y(t) = v\int_0^t e^{2z(\tau)}d\tau$ $z(t)$ comes from the separable differential equation $\frac{dz}{\pm\sqrt{1-u^2e^{-2z}-v^2e^{2z}}} = dt$, for $w \gtreqless 0$ whose solution is non-elementary function.
(2)	$u \neq 0$ $v \neq 0$ $w = 0$	$x(t) = ut$ $y(t) = vt$ $z(t) = 0$
(3)	$v = 0$ $0 < \lvert w\rvert = \sqrt{1-u^2} < 1$	$x(t) = u\frac{\sinh t}{\cosh t + w\sinh t}$ $y(t) = 0$ $z(t) = \ln(\cosh t + w\sinh t)$
(4)	$u = 0$ $0 < \lvert w\rvert = \sqrt{1-v^2} < 1$	$x(t) = 0$ $y(t) = v\frac{\sinh t}{\cosh t - w\sinh t}$ $z(t) = -\ln(\cosh t - w\sinh t)$
(5)	$u = 0$ $v = 0$ $\lvert w\rvert = 1$	$x(t) = 0$ $y(t) = 0$ $z(t) = \pm t$, for $w = \pm 1$

Table 1: *Table of geodesics in Sol geometry, depending on the initial velocity parameters* (u,v,w), $u^2+v^2+w^2=1$.

The forthcoming pictures try to visualize the most general cases (1) and (3). As we easily see the change $(u,v,w) \leftrightarrow (v,u,-w)$ leads to the isometry of the corresponding geodesic curves.

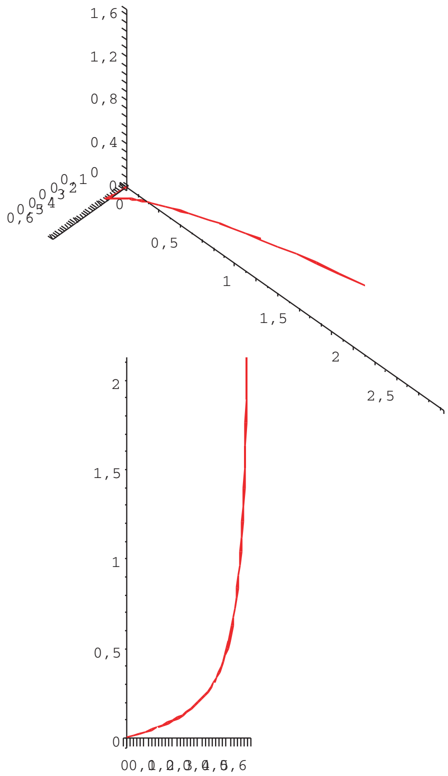

Fig. 1: *The approximate view of the most general geodesic curve with initial velocity parameters* $u = 0.9$ *and* $v = 0.25$ *in the parameter interval* $t \in [0,2]$. *The first picture shows the curve in a general view, the other from the direction of z-axis.*

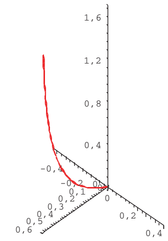

Fig. 2: *Geodesic curve starting in the [x,z] coordinate plane with* $u = 0.9$ *in a general view in the parameter interval* $t \in [0,2]$.

Clearly, the geodesic sphere unfortunately can not be expressed in a closed explicite form. We give approximations by plotting the endpoints of many geodesic curves (of the first type) with different initial unit velocities, according to geographic parameters

$$\begin{aligned} u &= \cos\vartheta\cos\varphi & -\pi \le \varphi \le \pi \\ v &= \cos\vartheta\sin\varphi & -\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2} \\ w &= \sin\vartheta \ . \end{aligned}$$

That means, if ϑ is fixed and φ varies, then the endpoints of geodesics describe an altitude circle. Similarly we get longitude half-circle for fixed φ. The following figures show our results.

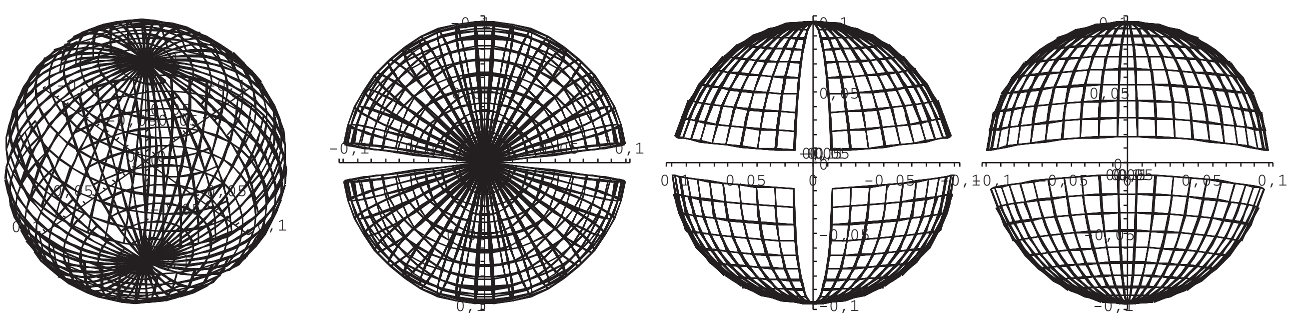

Fig. 3: *Geodesic sphere of radius 0.1. The first picture shows the sphere in a general view, then from the direction of axes z, y and x, respectively.*

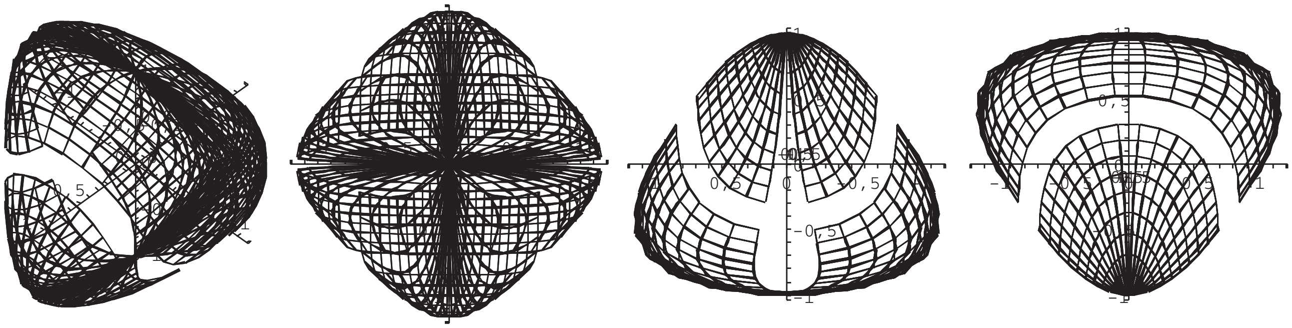

Fig. 4: *The same arrangement as in Fig. 4 but now the radius is 1.*

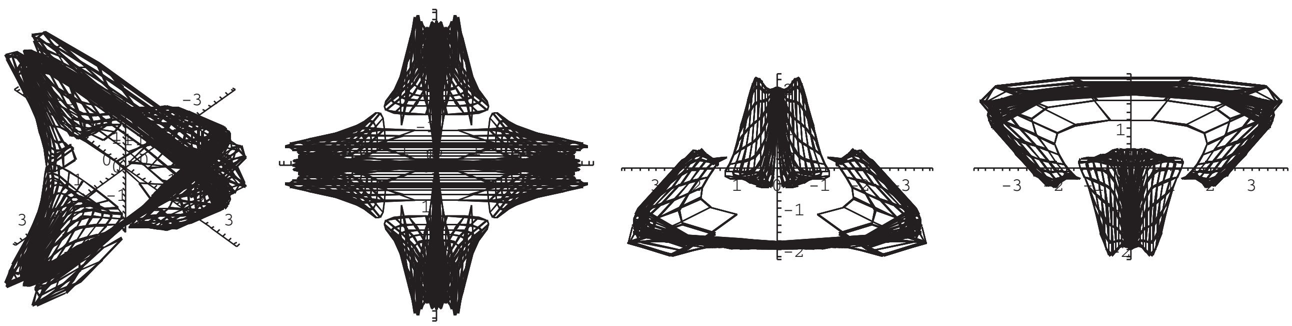

Fig. 5: *Geodesic sphere of radius 2.*

3 Translation curves and spheres

A Riemannian manifold with a transitive group of isometries is called homogeneous. In a homogeneous space there are postulated isometries, mapping each point to any point. Translations can be introduced in a natural way. Consider a unit vector at the origin. Translations, postulated at the beginning carry this vector to any point by its tangent mapping. If a curve $t \mapsto (x(t), y(t), z(t))$ has just the translated vector as tangent vector in each point, then the curve is called a *translation curve*. This assumption leads to a system of first order differential equations, thus translation curves are simpler than geodesics and differ from them in most cases (except in spaces of constant curvature).

From [3] we have already known the solution of the above defined system

$$\dot{x}(t) = ue^{-z(t)} \quad ,$$
$$\dot{y}(t) = ve^{z(t)} \quad ,$$
$$\dot{z}(t) = w \quad ,$$

of differential equation which holds for a curve starting at the origin in direction (u, v, w):

$$x(t) = -\frac{u}{w}\left(e^{-wt} - 1\right) \quad .$$
$$y(t) = \frac{v}{w}\left(e^{wt} - 1\right) \quad ,$$
$$z(t) = wt \quad ,$$

.

In the following -as illustration- we show how a translation curve looks like.

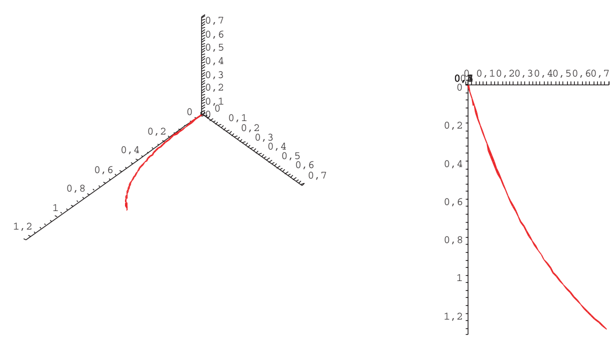

Fig. 6: *Translation curve with the same initial velocity parameters as for geodesics above, (*$u = 0.9$ *and* $v = 0.25$*) in the parameter interval* $t \in [0, 2]$*. The first picture shows the curve in a general view, the other from the direction of z-axis.*

With unit velocity translation curves we can define the *translation sphere* of radius r with centre in the origin of usual longitude and altidude parameters φ and ϑ, respectively ([3]):

$$u = \cos\vartheta\cos\varphi \qquad -\pi \leq \varphi \leq \pi$$
$$v = \cos\vartheta\sin\varphi \qquad -\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2}$$
$$w = \sin\vartheta \ ;$$

$$x(\vartheta,\varphi) = -\cot\vartheta\cos\varphi\left(e^{-r\sin\vartheta} - 1\right)$$
$$y(\vartheta,\varphi) = \cot\vartheta\sin\varphi\left(e^{r\sin\vartheta} - 1\right)$$
$$z(\vartheta,\varphi) = r\sin\vartheta \ .$$

As illustrations we give the following nice pictures.

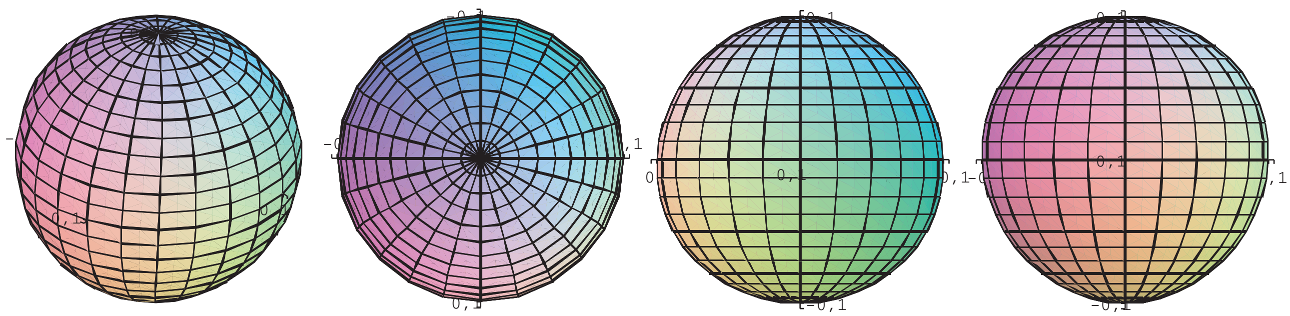

Fig. 7: *Translation sphere of radius 0.1. The first picture shows the sphere in a general view, then from the direction of axes z, y and x, respectively.*

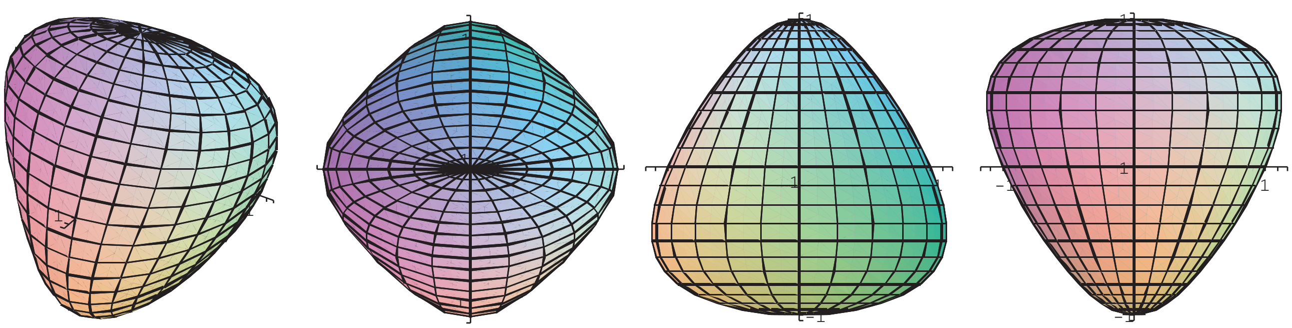

Fig. 8: *The same arrangement as above but now the radius is 1.*

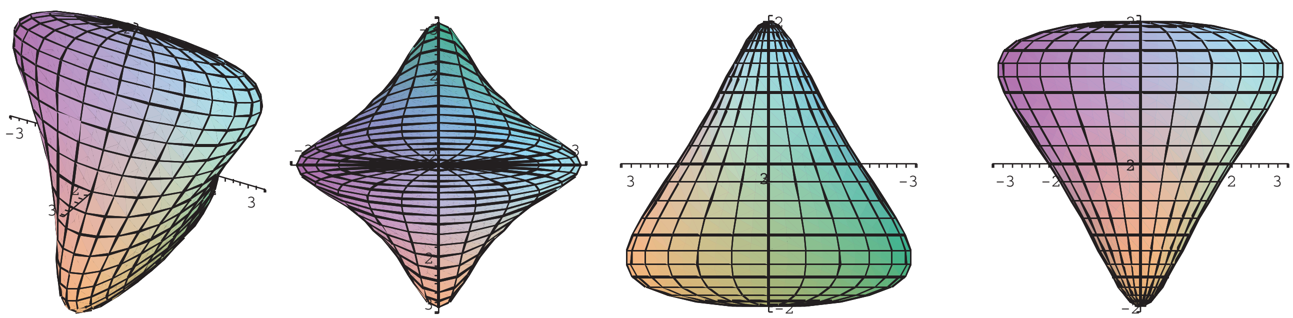

Fig. 9: *Translation sphere of radius 2.*

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