Original scientific paper
Accepted 29.12.2007.

Radimir Viher

On the Multiple Roots of the 4th Degree Polynomial

On the Multiple Roots of the 4th Degree Polynomial

ABSTRACT

In this article we investigate the connection between the multiple roots of the 4th degree polinomial $P_4(x)$ and its Descartes's cubic resolvent $P_3(x)$. The multiple roots of $P_4(x)$ are classified according to the position of all roots of the corresponding $P_3(x)$. Seven types are obtained.

Key words: 4th degree polynomial, Descartes's cubic resolvent, types of roots

MSC 2000: 14H45

O višestrukim korijenima polinoma 4. stupnja

SAŽETAK

U članku se istražuje veza između višestrukih korijena polinoma 4. stupnja $P_4(x)$ i njegove Descartesove kubne rezolvnete $P_3(x)$. Višestruki korijeni polinoma $P_4(x)$ razvrstani su ovisno o položaju svih korijena pripadne $P_3(x)$. Dobiveno je sedam tipova.

Ključne riječi: polinom 4. stupnja, Descartesova kubna rezolventa, tipovi korijena

In the previous article (see [6]), it is shown that we get the Descartes's cubic resolvent of the reduced polynomial of the fourth degree

$$P_4(x) \equiv x^4 + a_2x^2 + a_1x + a_0 \ (a_i \in \mathbf{R}, i = 0,1,2), \tag{1}$$

by factorization of $P_4(x)$

$$x^4 + a_2x^2 + a_1x + a_0 \equiv (x^2 + Ax + B)(x^2 + Cx + D) \tag{2}$$

and then seek the equation for deriving the value of A. By multiplying those two quadratic polynomials on the right side of (2) and then equating the coefficients of the same powers of x we get the following system of four equations with four unknowns

$$\begin{array}{ccccccc} A & + & C & & & = & 0 \\ AC & + & B & + & D & = & a_2 \\ AD & + & BC & & & = & a_1 \\ BD & & & & & = & a_0. \end{array} \tag{3}$$

When we solve this system we obtain the following equation (Descartes's cubic resolvent)

$$P_3(t) \equiv t^3 + 2a_2t^2 + (a_2^2 - 4a_0)t - a_1^2 = 0, \tag{4}$$

where $t = A^2$. Further in the above mentioned article there are the theorems about correspondences between the types of the roots of $P_3(t)$ and $P_4(x)$, and about characterizations of those types of roots of $P_3(t)$ formulated and proved. For the sake of further results we shall repeat the main definitions and formulations of those two theorems. As the free member of $P_3(t)$ is $-a_1^2$ and the coefficient of the greatest power of t is 1 we have three main possibilities for the types of roots of $P_3(t)$.

In the **first case**, $P_3(t)$ has only one real non-negative root and two conjugate complex roots or one real non-negative root and one real negative double root.

In the **second case**, $P_3(t)$ has one real non-negative root and two different real non-positive roots (the case of double root at zero is included in this case).

In the **third case**, $P_3(t)$ has three real non-negative roots (the cases of double and triple roots are included in this case).

Now we shall give the formulations of the theorem 1 and the theorem 2 of [6].

Theorem 1.

$$\begin{aligned} &\textit{1st case} &\iff& \; P_4(x) \textit{ has two real and two complex roots}\\ &\textit{2nd case} &\iff& \; P_4(x) \textit{ has only complex roots}\\ &\textit{3rd case} &\iff& \; P_4(x) \textit{ has only real roots}. \end{aligned}$$

Theorem 2.

$$\begin{aligned} &\textit{1st case} &\iff& \; D_1 > 0 \textit{ or } (D_1 = 0 \textit{ and } (a_2^2 - 4a_0 < 0 \textit{ or } (a_2^2 - 4a_0 > 0 \textit{ and } a_2 > 0)))\\ & & & \; \textit{or } (D_1 = 0 \textit{ and } a_2^2 - 4a_0 = 0 \textit{ and } a_2 > 0 \textit{ and } a_1 \neq 0)\\ &\textit{2nd case} &\iff& \; (D_1 < 0 \textit{ and } (a_2^2 - 4a_0 < 0 \textit{ or } (a_2^2 - 4a_0 \geq 0 \textit{ and } a_2 > 0)))\\ & & & \; \textit{or } (a_1 = 0 \textit{ and } a_2^2 - 4a_0 = 0 \textit{ and } a_2 > 0)\\ &\textit{3rd case} &\iff& \; D_1 \leq 0 \textit{ and } a_2^2 - 4a_0 \geq 0 \textit{ and } a_2 \leq 0. \end{aligned}$$

We get the quantity D_1 by using substitution $t = z - 2a_2/3$ in $P_3(t)$ that reduces it to

$$z^3 + pz + q = 0, \tag{5}$$

with

$$\begin{aligned} p &= -4a_0 - \frac{1}{3}a_2^2\\ q &= \frac{8}{3}a_0a_2 - a_1^2 - \frac{2}{27}a_2^3, \end{aligned} \tag{6}$$

and finally

$$D_1 = \frac{q^2}{4} + \frac{p^3}{27}. \tag{7}$$

This is a known procedure that leads to the Cardano's formula (see [3]). Before we formulate and prove the theorem about multiplicity correspondences between $P_4(x)$ and $P_3(t)$ we will show from which part of the theorem 1 proof comes the first indication for such theorem. In the proof of the "only if" part of the first statement we should first apply the factorization theorem for $P_4(x)$, so we get

$$P_4(x) = (x - x_1)(x - x_2)(x - a - bi)(x - a + bi). \tag{8}$$

Since the coefficient of the third power of x in $P_4(x)$ is zero, we obtain the following important relation

$$x_1 + x_2 + 2a = 0. \tag{9}$$

By using the (8) we can represent the $P_4(x)$ as a product of two quadratic polynomials in three different ways. In every such representation, the second power of the coefficient of x (no matter which one because they differ only in sign) is a root of $P_3(t)$ (see [6]). Thus, by using (8) we can find all roots of $P_3(t)$ but we have to distinguish two different cases. In the first case we suppose that $x_1 = x_2$, and together with (9) we get $x_1 + a = x_2 + a = 0$. Finally, from this one and from the three representations of (8) as product of two quadratic polynomials (by taking the second power of a coefficient of x in every such representation) we obtain

$$t_1 = 4a^2 \geq 0; \quad t_2 = t_3 = -b^2 < 0. \tag{10}$$

Hence, in this case the multiplicity of the real roots of $P_4(x)$ implies the same degree of the multiplicity of the real negative root of $P_3(t)$. If $x_1 \neq x_2$, it can be shown (see [6]) that there is no multiplicity of roots of $P_3(t)$ (because in this case t_2 and t_3 are conjugate complex numbers). Further on, we will show that this correspondence among the multiplicity of the roots of $P_4(x)$ and $P_3(t)$ is not a random event. For this purpose we shall divide all the possibilities of double and triple multiplicities of $P_3(t)$ on seven cases and in all those cases we will find the corresponding multiplicity of $P_4(x)$. It is time to look at the following seven figures and to analyze every one of them.

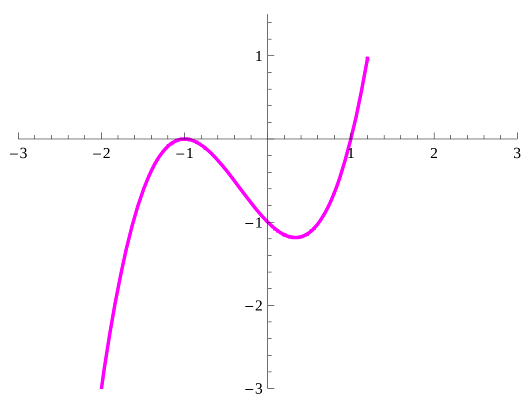

Fig. 1 a: 1st case

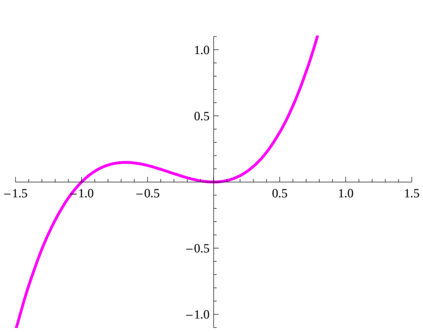

Fig. 1 b: 2nd case

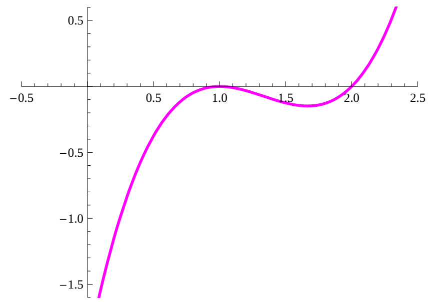

Fig. 1 c: 3rd case

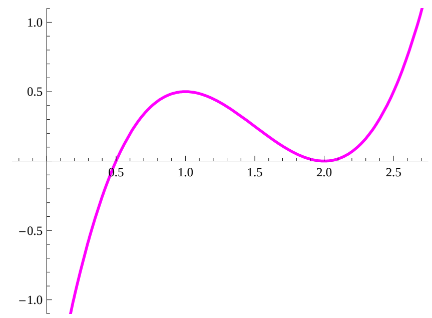

Fig. 1 d: 4th case

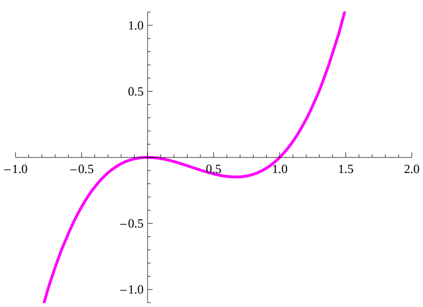

Fig. 1 e: 5th case

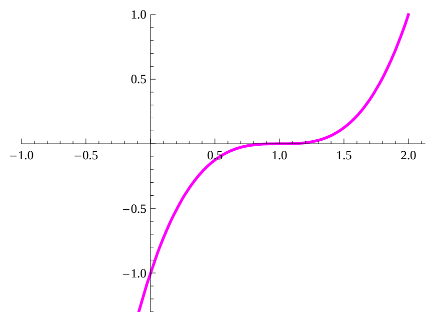

Fig. 1 f: 6th case

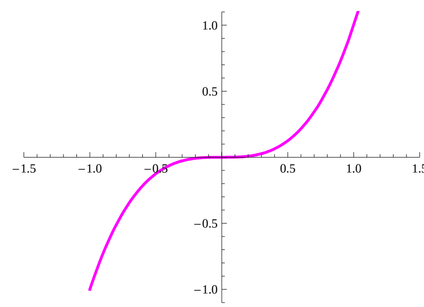

Fig. 1 g: 7th case

In the **1st case** (Fig. 1 a), $P_3(t)$ has a real negative double root and a real non-negative single root.

In the **2nd case** (Fig. 1 b), $P_3(t)$ has a zero as double root and a real negative single root.

In the **3rd case** (Fig. 1 c), $P_3(t)$ has a real positive double root which is smaller than the real positive single root.

In the **4th case** (Fig. 1 d), $P_3(t)$ has a real positive double root which is greater than the real non-negative single root.

In the **5th case** (Fig. 1 e), $P_3(t)$ has a zero as double root and a real positive single root.

In the **6th case** (Fig. 1 f), $P_3(t)$ has a real positive triple root.

In the **7th case** (Fig. 1 g), $P_3(t)$ has a zero as a triple root.

Now we can formulate and prove the main theorem.

Theorem 3.

1st case $\Longleftrightarrow$ $P_4(x)$ *has two complex roots and one double real root.*

2nd case $\Longleftrightarrow$ $P_4(x) = (x^2 + b^2)^2 \quad (b \neq 0)$

3rd case $\Longleftrightarrow$ $P_4(x)$ *has three different real roots, but among them is only one double root and the other two single roots are on the same side of the double one.*

4th case $\Longleftrightarrow$ $P_4(x)$ *has three different real roots, but among them is only one double root and the other two single roots are on the opposite sides of the double one.*

5th case $\Longleftrightarrow$ $P_4(x)$ *has two double real roots.*

6th case $\Longleftrightarrow$ $P_4(x)$ *has two different real roots and one of them is a triple root.*

7th case $\Longleftrightarrow$ $P_4(x)$ *has only one fourfold real root which is zero.*

Proof: At the bargaining we shall do some general considerations. It is known (see [7]) that the roots of $P_4(x)$ and $P_3(t)$ are connected by the following relations

$$\begin{aligned} t_1 &= -(x_1+x_2)(x_3+x_4)\\ t_2 &= -(x_1+x_3)(x_2+x_4)\\ t_3 &= -(x_1+x_4)(x_2+x_3). \end{aligned} \tag{11}$$

In the further considerations we shall use also the first Vieta's formula for the roots of $P_4(x)$ $x_1+x_2+x_3+x_4=0$ (because the coefficient of the third power of x is zero). From these formulas we can get

$$\begin{aligned} t_1 &= (x_1+x_2)^2 = (x_3+x_4)^2\\ t_2 &= (x_1+x_3)^2 = (x_2+x_4)^2\\ t_3 &= (x_1+x_4)^2 = (x_2+x_3)^2. \end{aligned} \tag{12}$$

Now we shall consider the case when two roots of $P_4(x)$ are real and the other two roots are complex. Let $x_3 = a+bi$ and $x_4 = a-bi$ $(b \neq 0)$. Then $t_2 = (x_1+x_3)^2 = (x_1+a+bi)^2$ must be a real number (because t_i for $i=1,2,3$ are always real numbers in all seven cases) but that is only possible iff $x_1+a=0$ respectively $x_1+x_2+x_3+x_4 = x_1+a+x_2+a = x_2+a = 0$ thus $x_1 = x_2 = -a$. From (12) immediately follows $t_2 = t_3 = -b^2 < 0$. In the case when all four roots of $P_4(x)$ are complex numbers, let $x_1 = a+bi, x_2 = a-bi, x_3 = c+di, x_4 = c-di$, then from $x_1+x_2+x_3+x_4 = 2a+2c = 0$ we get $a+c=0$. Hence from (12) it follows $t_2 = -(b+d)^2$ and $t_3 = -(b-d)^2$. From the fact that both numbers b and d are different from zero we conclude that $t_2 \neq t_3$ and at least one of them is less than zero. If a and c are both different from zero, then it follows from (12) $t_1 = 4a^2 > 0$, which means that all three roots of $P_3(t)$ are mutually different. If we want that at least two roots of $P_3(t)$ be the same, then it is necessary and sufficient that $a=c=0$ and $b=d$ or $b=-d$, which implies $x_1 = bi$, $x_2 = -bi$, $x_3 = bi$, $x_4 = -bi$. According to this all four roots are purely imaginary with two equal pairs. On the basis of all these considerations we conclude that

$$t_i \geq 0 \quad (i=1,2,3) \quad \Longleftrightarrow \quad x_i \in \mathbf{R} \quad (i=1,2,3,4). \tag{13}$$

So we proved the first two statements of our theorem and in the remaining five statements are all four roots of $P_4(x)$ only real numbers. From (12) and from $x_1+x_2+x_3+x_4=0$ we get

$$t_1 = t_2 < t_3 \quad \Longleftrightarrow \quad \begin{aligned} (x_2-x_3)(x_1-x_4) &= 0\\ (x_3-x_4)(x_1-x_2) &< 0\\ (x_2-x_4)(x_1-x_3) &< 0. \end{aligned} \tag{14}$$

So there are two possibilities

$$\begin{array}{lllll} I & x_1 = x_4 & \text{and} & (x_2,x_3 < x_4 \quad \text{or} \quad x_2,x_3 > x_4)\\ II & x_2 = x_3 & \text{and} & (x_1,x_4 < x_2 \quad \text{or} \quad x_1,x_4 > x_2). \end{array} \tag{15}$$

The case $x_1 = x_4$ and $x_2 = x_3$ is impossible because from $t_1 = -(x_1+x_2)(x_3+x_4) = -(x_1+x_2)^2 \leq 0$ we get a contradiction. Analogously in the case of fourth statement we get

$$t_1 < t_2 = t_3 \quad \Longleftrightarrow \quad \begin{aligned} (x_3-x_4)(x_1-x_2) &= 0\\ (x_2-x_3)(x_1-x_4) &< 0\\ (x_2-x_4)(x_1-x_3) &< 0. \end{aligned} \tag{16}$$

So there are two possibilities

$$\begin{array}{lllll} I & x_1 = x_2 & \text{and} & (x_3 < x_1 < x_4 \quad \text{or} \quad x_4 < x_1 < x_3)\\ II & x_3 = x_4 & \text{and} & (x_1 < x_3 < x_2 \quad \text{or} \quad x_2 < x_3 < x_1). \end{array} \tag{17}$$

The case $x_1 = x_2$ and $x_3 = x_4$ is impossible because from $t_2 = -(x_1+x_3)(x_2+x_4) = -(x_1+x_3)^2 \leq 0$ we get a contradiction. In the case of 5th statement we get again using (12) and $x_1+x_2+x_3+x_4=0$

$$0 = t_1 = t_2 < t_3 \quad \Longleftrightarrow \quad x_1 = -x_2 = -x_3 = x_4 \neq 0. \tag{18}$$

In the case of 6th statements we get analogously

$$t_1 = t_2 = t_3 \quad \Longleftrightarrow \quad \begin{aligned} (x_2-x_3)(x_1-x_4) &= 0\\ (x_3-x_4)(x_1-x_2) &= 0\\ (x_2-x_4)(x_1-x_3) &= 0 \end{aligned} \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \begin{aligned} x_1 &= x_2 = x_3\\ x_1 &= x_2 = x_4\\ x_1 &= x_3 = x_4\\ x_2 &= x_3 = x_4. \end{aligned} \tag{19}$$

If we supposed $x_1 = x_2 = x_3 = x_4$ we get that all four roots are equal to zero, which is a contradiction. In the case of 7th case by considerations of previous case we easily get $x_1 = x_2 = x_3 = x_4 = 0$. □

Remark. *As we have noted before, the proof of the "first case" in the theorem 1 is an indication for the existence of the theorem 3. Yet one indication for the theorem 3 is a fact that the discriminants of $P_4(x)$ and $P_3(t)$ are equal (see [7]).*

It remains only to formulate and prove the theorem about the characterizations of all those seven cases.

Theorem 4.

$$\begin{array}{lcl} \textit{1st case} & \iff & a_2^2+12a_0>0 \quad \textit{and} \quad -2a_2<\sqrt{a_2^2+12a_0} \quad \textit{and} \quad 2(a_2^2+12a_0)^{\frac{3}{2}}=2a_2^3-72a_2a_0+27a_1^2 \\ \textit{2nd case} & \iff & a_1=0 \quad \textit{and} \quad a_2^2-4a_0=0 \quad \textit{and} \quad a_2>0 \\ \textit{3rd case} & \iff & a_2<0 \quad \textit{and} \quad a_2^2-4a_0>0 \quad \textit{and} \quad a_2^2+12a_0>0 \quad \textit{and} \quad 2(a_2^2+12a_0)^{\frac{3}{2}}=2a_2^3-72a_2a_0+27a_1^2 \\ \textit{4th case} & \iff & a_2<0 \quad \textit{and} \quad a_2^2-4a_0>0 \quad \textit{and} \quad a_2^2+12a_0>0 \quad \textit{and} \quad 2(a_2^2+12a_0)^{\frac{3}{2}}=-2a_2^3+72a_2a_0-27a_1^2 \\ \textit{5th case} & \iff & a_1=0 \quad \textit{and} \quad a_2^2-4a_0=0 \quad \textit{and} \quad a_2<0 \\ \textit{6th case} & \iff & a_2^2+12a_0=0 \quad \textit{and} \quad 8a_2^3+27a_1^2=0 \quad \textit{and} \quad a_2<0 \\ \textit{7th case} & \iff & a_0=a_1=a_2=0. \end{array} \tag{20}$$

Proof:

$$\begin{aligned} P_3(t)&=t^3+2a_2t^2+(a_2^2-4a_0)t-a_1^2\\ P_3'(t)&=3t^2+4a_2t+a_2^2-4a_0\\ P_3''(t)&=6t+4a_2. \end{aligned} \tag{21}$$

In the 1st case $P_3'(t)$ should have two different real roots and the smaller one should be double real root of $P_3(t)$. For $P_3'(t)$ to have different real roots, the necessary and sufficient condition is

$$a_2^2+12a_0>0. \tag{22}$$

If condition (22) is satisfied then the smaller real root of $P_3'(t)$ is

$$t_1=\frac{-2a_2-\sqrt{a_2^2+12a_0}}{3}. \tag{23}$$

The condition necessary and sufficient for $P_3(t_1)=0$ is

$$2(a_2^2+12a_0)^{\frac{3}{2}}=2a_2^3-72a_2a_0+27a_1^2. \tag{24}$$

Finally, the condition for t_1 to be a negative real number is equivalent to

$$-2a_2<\sqrt{a_2^2+12a_0}. \tag{25}$$

In the 2nd case $t=0$ is a double real root of $P_3(t)$ which is evidently equivalent to

$$a_1=0 \quad \text{and} \quad a_2^2-4a_0=0. \tag{26}$$

As the second single real root of $P_3(t)$ is negative, we easily conclude that

$$a_2>0. \tag{27}$$

In the 3rd case $P_3'(t)$ has two different positive real roots which is equivalent to

$$\begin{aligned} &P_3'(0)=a_2^2-4a_0>0 \quad \text{and}\\ &P_3''(0)=4a_2<0 \quad \text{and}\\ &a_2^2+12a_0>0. \end{aligned} \tag{28}$$

The last condition is obtained by analogous reasoning like in the 1st case. Finally, analogously as in the 1st case, it must be $P_3(t_1)=0$ where t_1 is given by (23), so we get the relation (24) again.

In the 4th case we get the conditions (28) using the same reasoning like in the 3rd case. But now $P_3(t_2)=0$ where t_2 is the greater real root of $P_3'(t)$ i. e.

$$t_2=\frac{-2a_2+\sqrt{a_2^2+12a_0}}{3}. \tag{29}$$

Condition $P_3(t_2)=0$ is equivalent to the following condition

$$2(a_2^2+12a_0)^{\frac{3}{2}}=-2a_2^3+72a_2a_0-27a_1^2. \tag{30}$$

In the 5th case, analogously as in the 2nd case, $t=0$ must be the double real root of $P_3(t)$ which is equivalent to the conditions (26). But now, since the second single real root of $P_3(t)$ is positive, we conclude easily that

$$a_2<0. \tag{31}$$

In the 6th case

$$P_3(t)=0, \quad P_3'(t)=0, \quad P_3''(t)=0 \tag{32}$$

must be for the same value of t. From $P_3''(t)=0$ we easily get

$$t=-\frac{2}{3}a_2. \tag{33}$$

From (33) we get immediately

$$a_2<0, \tag{34}$$

and

$$P_3'\left(-\frac{2}{3}a_2\right)=0 \quad \text{and} \quad P_3\left(-\frac{2}{3}a_2\right)=0, \tag{35}$$

which is equivalent to

$$a_2^2+12a_0=0 \quad \text{and} \quad 8a_2^3+27a_1^2=0. \tag{36}$$

In the 7th case it should be

$$P_3(0)=0; \quad P_3'(0)=0; \quad P_3''(0)=0, \tag{37}$$

which is equivalent to

$$a_0=0; \quad a_1=0; \quad a_2=0. \tag{38}$$

□

Now, there is question left about the conditions that give the answer in the 3rd and the 6th case to the question on which side of the double (triple) root are two single roots (one single root). That problem is solved in the next lemma.

Lemma 5. *Let $P_3(t)$ have a real positive double root and a real single root which is greater than that double one. Then*

$$x_{1,2}=\frac{a_1(a_2^2+12a_0)}{8a_0a_2-2a_2^3-9a_1^2} \tag{39}$$

is double root of $P_4(x)$ and

$$a_1>0\,(<0) \iff x_3 \text{ and } x_4 \text{ are on the left (right) side of } x_{1,2}. \tag{40}$$

Let $P_3(t)$ have a positive triple root. Then

$$x_{1,2,3}=-\frac{8}{3}\frac{a_0}{a_1} \tag{41}$$

is a triple root of $P_4(x)$ and

$$a_1>0\,(<0) \iff x_4 \text{ is on the left (right) side of } x_{1,2,3}. \tag{42}$$

Proof: From the theorem 3 it follows that $P_4(x)$ has only one double real root. We will find that double root by eliminating the members with higher powers of x between the following two equations

$$\begin{aligned} &P_4(x)\equiv x^4+a_2x^2+a_1x+a_0=0\\ &P_4'(x)\equiv 4x^3+2a_2x+a_1=0. \end{aligned} \tag{43}$$

By eliminating the member with x^4 between these two equations we get

$$2a_2x^2+3a_1x+4a_0=0. \tag{44}$$

By eliminating the member with x^3 between (44) and the second equations of (43) we get

$$6a_1x^2+(8a_0-2a_2^2)x-a_1a_2=0. \tag{45}$$

Finally, by eliminating the member with x^2 between (44) and (45) we get

$$(8a_0a_2-2a_2^3-9a_1^2)x-a_1a_2^2-12a_0a_1=0. \tag{46}$$

From the equation (46) the formula (39) follows. Using the relations which characterize the 3rd case in the theorem 4 we will prove that

$$8a_0a_2-2a_2^3-9a_1^2>0. \tag{47}$$

So we get

$$\begin{aligned} &a_2^2-4a_0>0 \implies 12a_2^2-48a_0>0 \implies\\ &16a_2^2>4(a_2^2+12a_0) \implies\\ &\implies 2(a_2^2+12a_0)^{\frac{3}{2}}<-4a_2(a_2^2+12a_0) \implies\\ &\implies 2(a_2^2+12a_0)^{\frac{3}{2}}=27a_1^2+2a_2^3-72a_0a_2<\\ &<-4a_2(a_2^2+12a_0) \implies 8a_0a_2-2a_2^3-9a_1^2>0. \end{aligned} \tag{48}$$

From $a_2^2+12a_0>0$ and (47) it follows that the sign of $x_{1,2}$ is the same as of a_1. It remains only to prove that a_1 cannot be zero. Suppose conversely that a_1 is zero. Then from

$$x_1=x_2=0 \quad \text{and} \quad x_1+x_2+x_3+x_4=0 \tag{49}$$

we conclude that $x_4=-x_3$ which is in contradiction with the 3rd case (because x_3 and x_4 are on the opposite sides of double root zero). Now, if the sign of $x_{1,2}$ is positive $(a_1>0)$ then from

$$x_{1,2}=-\frac{x_3+x_4}{2} \tag{50}$$

we conclude that x_3 and x_4 lie on the left side of $x_{1,2}$. Completely analogously we reason in the case when $x_{1,2}$ is negative $(a_1<0)$.

If $P_3(t)$ has positive triple root then according to theorem 3, $P_4(x)$ has a triple real root. We will get it by eliminating the members of higher powers of x among the following three equations

$$\begin{aligned} P_4(x)&\equiv x^4+a_2x^2+a_1x+a_0 &&=0\\ P_4'(x)&\equiv 4x^3+2a_2x+a_1 &&=0\\ P_4''(x)&\equiv 12x^2+2a_2 &&=0. \end{aligned} \tag{51}$$

From the first two equations we get

$$2a_2x^2+3a_1x+4a_0=0. \tag{52}$$

From the (52) and the last equation of (51) we get finally

$$9a_1x+12a_0-a_2^2=0. \tag{53}$$

From (53) it follows

$$x_{1,2,3} = \frac{a_2^2 - 12a_0}{9a_1}. \tag{54}$$

Using the relations which characterize the 6th case in the theorem 4 we easily get that a_1 cannot be zero and secondly that

$$x_{1,2,3} = -\frac{8}{3}\frac{a_0}{a_1}. \tag{55}$$

From relations $a_2^2 + 12a_0 = 0$ and $a_2 < 0$ it follows $a_0 < 0$. From

$$x_4 = -3x_{1,2,3} \quad \text{and} \quad a_0 < 0 \tag{56}$$

we easily get (42). □

References

[1] W. W. Rause Ball, *A Short Account of the History of Mathematics*, Dover Publications, New York, 1960.

[2] A. B. Basset, *An Elementary Treatise on Cubic and Quartic Curves*, Deighton Bell and Co., Cambridge, 1901.

[3] B. Pavković, D. Veljan, *Elementarna matematika I*, Tehnička knjiga, Zagreb, 1992.

[4] J. Plemelj, *Algebra in teorija števil*, Slovenska akademija znanosti in umetnosti, Ljubljana, 1962.

[5] V. Sedmak, *Uvod u algebru*, Sveučilište u Zagrebu, Zagreb, 1961.

[6] R. Viher, The Consequences of Descartes's Method for Factorization of 4th Degree Polynomial, KoG no. 5, Zagreb, 2001.

[7] S. Warner, *Modern Algebra*, Dover Publications, New York, 1990.

Radimir Viher
Faculty of Civil Engineering, University of Zagreb
Kačićeva 26, 10000 Zagreb, Croatia
e-mail: viher@master.grad.hr