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# Butterflies in the Isotropic Plane

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**ABSTRACT**

A real affine plane $A_2$ is called an isotropic plane $I_2$, if in $A_2$ a metric is induced by an absolute $\{f,F\}$, consisting of the line at infinity $f$ of $A_2$ and a point $F \in f$. In this paper the well-known Butterfly theorem has been adapted for the isotropic plane. For the theorem that we will further-on call an Isotropic butterfly theorem, four proofs are given.

**Key words:** isotropic plane, butterfly theorem

**MSC 2000:** 51N025

**Leptiri u izotropnoj ravnini**

**SAŽETAK**

Realna afina ravnina $A_2$ se naziva izotropnom ravninom $I_2$ ako je metrika u $A_2$ inducirana apsolutnom figurom $\{f,F\}$, koja se sastoji od neizmjerno dalekog pravca $f$ ravnine $A_2$ i točke $F \in f$. U ovom je radu poznati Leptirov teorem smješten u izotropnu ravninu. Za taj teorem, kojeg od sada nazivamo Izotropnim leptirovim teoremom, dana su četiri dokaza.

**Ključne riječi:** izotropna ravnina, leptirov teorem

## 1 Isotropic Plane

Let $P_2(\mathbf{R})$ be a real projective plane, $f$ a real line in $P_2$, and $A_2 = P_2 \backslash f$ the associated affine plane. The *isotropic plane* $I_2(\mathbf{R})$ is a real affine plane $A_2$ where the metric is introduced with a real line $f \subset P_2$ and a real point $F$ incidental with it. The ordered pair $\{f,F\}$, $F \in f$ is called *absolute figure* of the isotropic plane $I_2(\mathbf{R})$ ([3], [5]). In the affine model, where

$$x = x_1/x_0, \quad y = x_2/x_0, \tag{1}$$

the absolute figure is determined by the *absolute line* $f \equiv x_0 = 0$, and the *absolute point* $F$ (0:0:1). All projective transformations that are keeping the absolute figure fixed form a 5-parametric group

$$G_5 \begin{cases} \overline{x} = c_1 + c_4 x \\ \overline{y} = c_2 + c_3 x + c_5 y \end{cases}, \quad \begin{matrix} c_1, c_2, c_3, c_4, c_5 \in \mathbf{R} \\ \& \quad c_4 c_5 \neq 0. \end{matrix} \tag{2}$$

We call it *the group of similarities* of isotropic plane.

Defining in $I_2$ the usual metric quantities such as the distance between two points, the angle between two lines etc., we look for the subgroup of $G_5$ for those quantities to be invariant. In such a way one obtains the fundamental group of transformations that are the mappings of the form:

$$G_3 \begin{cases} \overline{x} = c_1 + x \\ \overline{y} = c_2 + c_3 x + y \end{cases}. \tag{3}$$

It is called *the motion group* of isotropic plane. Hence, the group of isotropic motions consists of translations and rotations, that is

$$\begin{cases} \overline{x} = c_1 + x \\ \overline{y} = c_2 + y \end{cases} \quad \text{and} \quad \begin{cases} \overline{x} = x \\ \overline{y} = c_3 x + y \end{cases}.$$

In the affine model, rotation is understood as stretching along the y-axis.

## 2 Terms of Elementary Geometry within $I_2$

We will first define some terms and point out some properties of triangles and circles in $I_2$ that are going to be used further on. The geometry of $I_2$ could be seen for example in Sachs [3], or Strubecker [5].

***Isotropic straight line, parallel points, isotropic distance, isotropic span***

All straight lines through the point $F$ are called *isotropic straight lines* (isotropic lines). All the other straight lines are simply called *straight lines*. Two points $A$, $B$ ($A \neq B$) are called *parallel* if they are incidental with the same isotropic line. For two no parallel points $A(a_1,a_2)$, $B(b_1,b_2)$, the *isotropic distance* is defined by $d(A,B) := b_1 - a_1$. Note that the isotropic distance is directed. For two parallel points $A(a_1,a_2)$, $B(b_1,b_2)$, $a_1 = b_1$, the quantity known as *isotropic spann* is defined by $s(A,B) := b_2 - a_2$. A straight line $p$ through two points $A$ and $B$ will be denoted by $p \equiv A \vee B$, or simply $p \equiv AB$.

***Invariants of a pair of straight lines***

Each no isotropic straight line $g \subset I_2$ can be written in the normal form $y = ux + v$, that is, in line coordinates, $g(u,v)$. For two straight lines $g_1(u_1,v_1)$, $g_2(u_2,v_2)$ the *isotropic angle*is defined by $\varphi = \angle(g_1,g_2) := u_2 - u_1$ . Note that the isotropic angle is directed as well. The Euclidean meaning of the isotropic angle can be understood from the affine model that is given in figure 1.

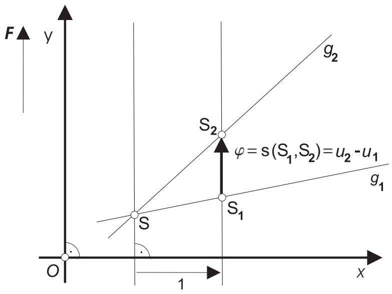


Fig. 1

For two parallel straight lines $g_1(u_1,v_1)$ , $g_2(u_1,v_2)$ there exists an isotropic invariant defined by $\varphi * (g_1,g_2) := v_2 - v_1$ (see fig. 2).

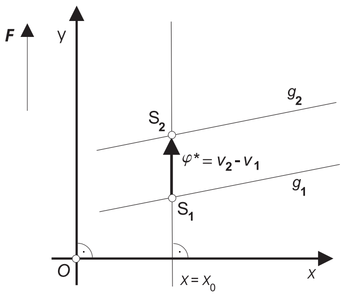


Fig. 2

***Isotropic normal***

An *isotropic normal* to the straight line $g(u,v)$ in the point $P(p_1,p_2)$ , $P \notin g$ is an isotropic line through $P$. Inversely holds as well, i.e. each straight line $g \subset I_2$ is a normal for each isotropic straight line. Denoting by $S$ the point of intersection of the isotropic normal in the point $P$ with the straight line $g$, the isotropic distance of the point $P$ from the line $g$ is given by $d(P,g) := s(S,P) = p_2 - s_2 = p_2 - up_1 - v$ (see fig. 3).

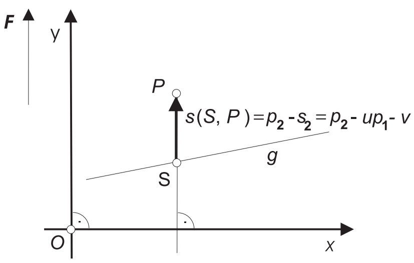


Fig. 3

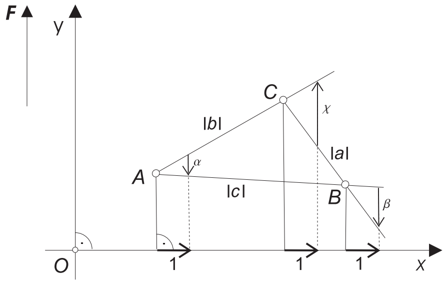


Fig. 4

***Triangles and circles***

Under a *triangle* in $I_2$ an ordered set of three no collinear points $\{A,B,C\}$ is understood. $A$, $B$, $C$ are called *vertices*, and $a := B \vee C$, $b := C \vee A$, $c := A \vee B$ *sides* of a triangle. A triangle is called *allowable* if no one of its sides is isotropic. In a allowable triangle the *lengths* of the sides are defined by $|a| := d(B,C)$, $|b| := d(C,A)$, $|c| := d(A,B)$, with $|a| \neq 0$, $|b| \neq 0$, $|c| \neq 0$. For the directed angles we have $\alpha := \angle(b,c) \neq 0$, $\beta := \angle(c,a) \neq 0$, $\gamma := \angle(a,b) \neq 0$ (see figure 4).

*Isotropic altitudes* $h_a$, $h_b$, $h_c$ associated with sides $a$, $b$, and $c$ are isotropic straight lines passing through the vertices $A$, $B$, $C$, i.e. normals to the sides $a$, $b$, and $c$. Their lengths are defined by $|h_a| := s(L(A),A)$, where $L(A) = a \cap h_a$, etc. The Euclidian meaning is given in figure 5.

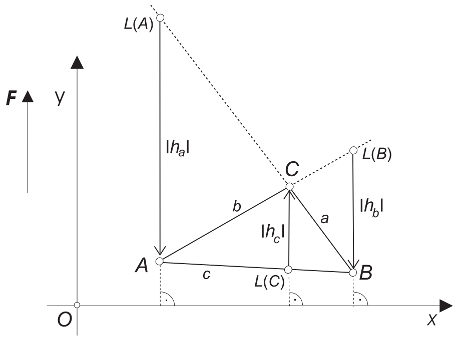


Fig. 5

An *isotropic circle* (*parabolic circle*) is a regular $2^{nd}$ order curve in $P_2(\mathbf{R})$ which touches the absolute line $f$ in the absolute point $F$. According to the group $G_3$ of motions of the isotropic plane there exists in $I_2$ a three parametric family of isotropic circles, given by $y = Rx^2 + \alpha x + \beta$, $R \neq 0$, $\alpha, \beta \in \mathbf{R}$. Using transformations from $G_3$, each isotropic circle can be reduced in the normal form $y = Rx^2$, $R \neq 0$. $R$ is a $G_3$ invariant and it is called the *isotropic radius* of the parabolic circle.

## 3 The Isotropic Butterfly Theorem

**Theorem 1 (Euclidean version)** *Let $M$ be the midpoint of a chord $PQ$ of the circle, through which two other chords $AB$ and $CD$ are drawn; $AD$ cuts $PQ$ at $X$ and $BC$ cuts $PQ$ at $Y$. $M$ is also the midpoint of $XY$.*

This theorem has been proved in a series of books and papers (e.g. [1], [2], [4]).

**Theorem 2 (Isotropic version)** *Let $M$ be the midpoint of a chord $\overrightarrow{PQ}$ of the parabolic circle, through which two other chords $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are drawn; $\overrightarrow{AD}$ cuts $\overrightarrow{PQ}$ at $X$ and $\overrightarrow{BC}$ cuts $\overrightarrow{PQ}$ at $Y$. $M$ is also the midpoint of $\overrightarrow{XY}$.*

**Proof 1**
The point coordinates are: $P(p_1,p_2)$, $Q(q_1,q_2)$, $M(m_1,m_2)$, $X(x_1,x_2)$, $Y(y_1,y_2)$, with $p_1 \neq q_1$, since $\overrightarrow{PQ}$ is a chord and as a such a no isotropic line, wherefrom we derive that $x_1 \neq y_1 \neq m_1$ must be fulfilled as well. Let us drop perpendiculars $h_1$, $h_2$ from $X$, and $g_1$, $g_2$ from $Y$ on $AB$ and $CD$. Let's also denote

$$\begin{array}{l} d(P,M) = d(M,Q) = |s|, \\ d(X,M) = |x|, \quad d(M,Y) = |y|, \end{array} \tag{4}$$

$$\begin{array}{ll} H_1 = h_1 \cap AM, & H_2 = h_2 \cap DM, \\ G_1 = g_1 \cap MB, & G_2 = g_2 \cap MC. \end{array} \tag{5}$$

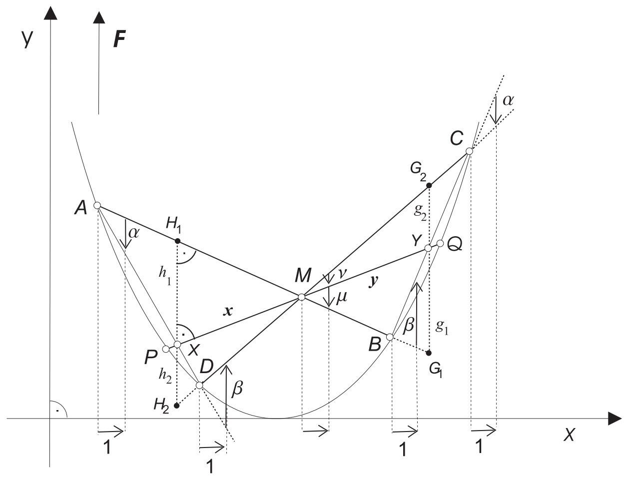


Fig. 6: The Isotropic butterfly theorem in the affine model

As first we need the following:

**Lemma 1** *Let $P$, $Q$, $P \neq Q$, be two points on a parabolic circle $k$, and $A \neq P$, $A \neq Q$, any other point on the same circle $k$. The isotropic angle $\varphi = \angle\left(\overrightarrow{PA},\overrightarrow{QA}\right)$ does not depend on the position of point $A$.*

The proof is given in [3, p. 32].

**Lemma 2** *The relations*

$$\frac{|a|}{\alpha} = \frac{|b|}{\beta} = \frac{|c|}{\chi}, \quad |h_a| = |c|\beta, \quad |h_b| = |a|\chi, \quad |h_c| = |b|\alpha$$

*hold for every allowable triangle.*

The proof is given in [3, p. 28].

**Lemma 3** *Let $k$ be a parabolic circle in $I_2$, a point $P \in I_2$, $P \notin k$, and $S_1$, $S_2$ two points of intersection of a no isotropic straight line $g$ through $P$ with $k$. The product $f(P) := d(P,S_1) \cdot d(P,S_2)$ doesn't depend of the line $g$, but only of $k$ and $P$.*

The proof is given in [3, p. 38].

Let's now continue the proof of the isotropic Butterfly theorem.
According to lemma 1,

$$\alpha = \angle\left(\overrightarrow{AB},\overrightarrow{AD}\right) = \alpha' = \angle\left(\overrightarrow{CB},\overrightarrow{CD}\right),$$

and

$$\beta = \angle\left(\overrightarrow{DA},\overrightarrow{DC}\right) = \beta' = \angle\left(\overrightarrow{BA},\overrightarrow{BC}\right). \tag{6}$$

We will also need

$$\mu = \angle\left(\overrightarrow{XM},\overrightarrow{MA}\right) = \mu' = \angle\left(\overrightarrow{YM},\overrightarrow{MB}\right),$$

and

$$\nu = \angle\left(\overrightarrow{DM},\overrightarrow{MX}\right) = \nu' = \angle\left(\overrightarrow{CM},\overrightarrow{MY}\right). \tag{7}$$

Let's apply furthermore lemma 2 on the following pairs of allowable triangles:
1st) $\triangle AXM$ & $\triangle MBY$, 2nd) $\triangle XDM$ & $\triangle MYC$,
3rd) $\triangle AXM$ & $\triangle MYC$, 4th) $\triangle XDM$ & $\triangle MBY$,
marking sides, angles and altitudes as given in figure 7.

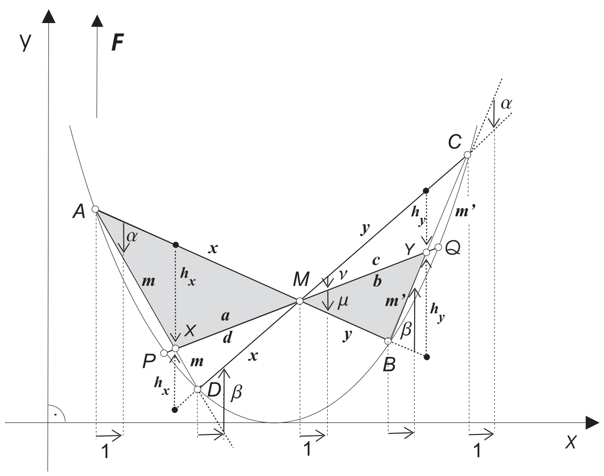


Fig. 7

1st) $\triangle AXM \;\Rightarrow\; \dfrac{|x|}{\angle\left(\overrightarrow{AX},\overrightarrow{XM}\right)}=\dfrac{|a|}{\alpha}=\dfrac{|m|}{\mu},$
$|h_x|=|a|\cdot\mu;$

$\triangle MBY \;\Rightarrow\; \dfrac{|y|}{\angle\left(\overrightarrow{BY},\overrightarrow{YM}\right)}=\dfrac{|m'|}{\mu}=\dfrac{|b|}{\beta},$
$|h_y|=|b|\cdot\mu;$

$\Rightarrow \dfrac{|h_x|}{|h_y|}=\dfrac{|a|}{|b|}$, and using marks from fig. 6 we get

$$\frac{|x|}{|y|}=\frac{|h_1|}{|g_1|}. \tag{8}$$

2nd) $\triangle XDM \;\Rightarrow\; \dfrac{|x|}{\angle\left(\overrightarrow{MX},\overrightarrow{XD}\right)}=\dfrac{|d|}{\beta}=\dfrac{|m|}{\nu},$
$|h_y|=|m|\cdot\beta=|d|\cdot\nu;$

$\triangle MYC \;\Rightarrow\; \dfrac{|y|}{\angle\left(\overrightarrow{MY},\overrightarrow{YC}\right)}=\dfrac{|c|}{\alpha}=\dfrac{|m'|}{\nu},$
$|h_y|=|m'|\cdot\alpha=|c|\cdot\nu;$

$\Rightarrow \dfrac{|h_x|}{|h_y|}=\dfrac{|d|}{|c|}$, and using marks from fig. 6 we have

$$\frac{|x|}{|y|}=\frac{|h_2|}{|g_2|}. \tag{9}$$

Analogously, for the third pair of triangles we get

$$\frac{|h_1|}{|g_2|}=\frac{d(A,X)}{d(Y,C)}. \tag{10}$$

Finally, for the fourth pair of triangles we have

$$\frac{|h_2|}{|g_1|}=\frac{d(X,D)}{d(B,Y)}. \tag{11}$$

From (4), (8), (9), (10), (11), and lemma 3 one computes

$$\begin{aligned}\frac{|x|^2}{|y|^2}&=\frac{|h_1|}{|g_1|}\cdot\frac{|h_2|}{|g_2|}=\frac{|h_1|}{|g_2|}\cdot\frac{|h_2|}{|g_1|}=\\&=\frac{d(A,X)}{d(Y,C)}\cdot\frac{d(X,D)}{d(B,Y)}=\frac{-d(X,A)\cdot d(X,D)}{-d(Y,C)\cdot d(Y,B)}=\\&=\frac{d(X,P)\cdot d(X,Q)}{d(Y,P)\cdot d(Y,Q)}=\frac{(p_1-x_1)(q_1-x_1)}{(p_1-y_1)(q_1-y_1)}=\\&=\frac{(p_1-m_1+m_1-x_1)(q_1-m_1+m_1-x_1)}{(p_1-m_1+m_1-y_1)(q_1-m_1+m_1-y_1)}=\\&=\frac{-(|s|-|x|)\ (|s|+|x|)}{-(|s|+|y|)\ (|s|-|y|)}=\frac{|s|^2-|x|^2}{|s|^2-|y|^2}.\end{aligned} \tag{12}$$

$$\frac{|x|^2}{|y|^2}=\frac{|s|^2-|x|^2}{|s|^2-|y|^2}\quad\Rightarrow\quad |x|^2=|y|^2\quad\Rightarrow\quad |x|=\pm|y|$$

The solution $|x|=-|y| \;\Rightarrow\; d(X,M)=-d(M,Y)=d(Y,M)$, wherefrom it follows that points $X$ and $Y$ are parallel points, which has been excluded earlier.
So, $|x|=|y| \;\Rightarrow\; d(X,M)=d(M,Y)$. □

**Proof 2**
Let's use the notation given in (4), that is, $d(P,M)=d(M,Q)=|s|$, $d(X,M)=|x|$, $d(M,Y)=|y|$, as well as (6) and (7) for the observed angles.

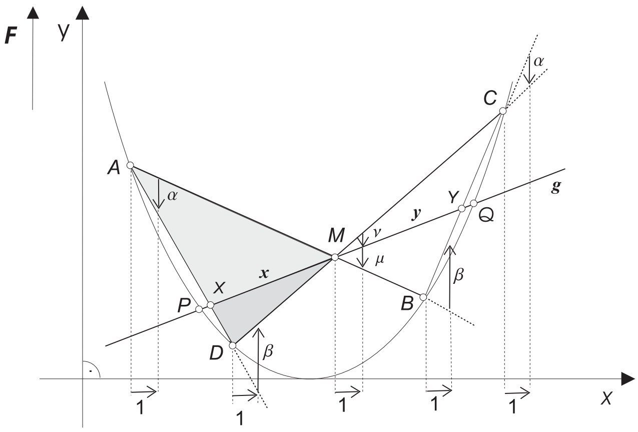


Fig. 8

From lemma 3, as shown in (12), we have

$$d(X,A)\cdot d(X,D)=d(X,P)\cdot d(X,Q),$$

$$d(X,P)\cdot d(X,Q)=-(|s|-|x|)\ (|s|+|x|)=|x|^2-|s|^2. \tag{13}$$

Lemma 2 applied on the allowable triangles $\triangle DMX$ and $\triangle AXM$ yields

$$\triangle DMX \Rightarrow \frac{d(X,D)}{\nu}=\frac{d(D,M)}{\angle\left(\overrightarrow{MX},\overrightarrow{XD}\right)}=\frac{d(M,X)}{\beta}$$

$$\Rightarrow \frac{d(X,D)}{\nu}=\frac{d(M,X)}{\beta} \tag{14}$$

$$\triangle AXM \Rightarrow \frac{d(A,X)}{\mu}=\frac{d(X,M)}{\alpha}=\frac{d(M,A)}{\angle\left(\overrightarrow{AX},\overrightarrow{XM}\right)}$$

$$\Rightarrow \frac{d(A,X)}{\mu}=\frac{d(X,M)}{\alpha}. \tag{15}$$

**Lemma 4** *The sum of the directed sides of an allowable triangle in $I_2$ equals zero; the sum of the directed angles of an allowable triangle in $I_2$ equals zero as well.*

The proof is given in [3, p. 22].

For the allowable triangle $\triangle ADM$, from lemma 4,

$$\nu+\mu+\alpha+\beta=0 \quad \Rightarrow \quad \beta=-(\nu+\mu+\alpha). \qquad (16)$$

Using (13)-(16) together, we obtain

$$d(X,A)\cdot d(X,D)=-d(X,M)\cdot\frac{\mu}{\alpha}\cdot d(M,X)\cdot\frac{\nu}{\beta}=$$

$$=|x|^2\frac{\nu\mu}{-\alpha(\nu+\mu+\alpha)}=|x|^2-|s|^2$$

$$\Rightarrow \quad |x|^2\left(1+\frac{\nu\mu}{\alpha(\nu+\mu+\alpha)}\right)=|s|^2$$

$$\Rightarrow \quad |x|^2=\frac{|s|^2\left[\alpha(\nu+\mu+\alpha)\right]}{\nu\mu+\alpha(\nu+\mu+\alpha)}. \qquad (17)$$

Following the same procedure ((13)-(16)) for the segment $|y|=d(M,Y)$, due to the symmetry in $\nu$ and $\mu$ in the latter expression, we'll get exactly same result. So, $|x|^2=|y|^2$, that is $|x|=\pm|y|$, and following the conclusion from proof 1, $|x|=|y| \Rightarrow d(X,M)=d(M,Y)$. □

**Proof 3**

The proof is based on the following:

**Lemma 5** *If in two allowable triangles in $I_2$ a directed angle of one is equal to a directed angle of the other, then the areas of the triangles are in the same ratio as the products of the sides composing the equal angles.*

**Proof** According [3, p. 26] the isotropic area of an allowable triangle $ABC$, $A(a_1,a_2)$, $B(b_1,b_2)$, and $C(c_1,c_2)$ is given by

$$F_{ABC}=\frac{1}{2}\begin{vmatrix}1&1&1\\a_1&b_1&c_1\\a_2&b_2&c_2\end{vmatrix}.$$

Let's mark the directed angles as given before in (6) and (7) (see figure 6), and let's observe the allowable triangles $AXM$ and $MYC$ (figure 9).

Lemma 1 yields that $\alpha=\angle\left(\overrightarrow{MA},\overrightarrow{AX}\right)=\alpha'=\angle\left(\overrightarrow{YC},\overrightarrow{CM}\right)$, hence, we have to proof the equality:

$$\frac{F_{AXM}}{F_{MYC}}=\frac{d(M,A)\cdot d(A,X)}{d(Y,C)\cdot d(C,M)}. \qquad (18)$$

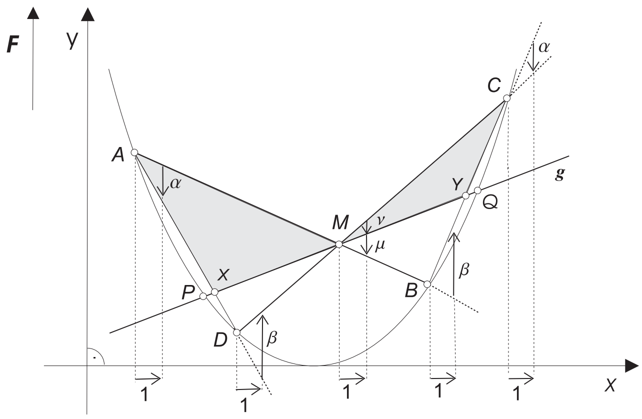


Fig. 9

For the points $A(a_1,a_2)$, $C(c_1,c_2)$, $M(m_1,m_2)$, $X(x_1,x_2)$ and $Y(y_1,y_2)$, the isotropic areas of the triangles are given by

$$F_{AXM}=\frac{1}{2}\begin{vmatrix}1&1&1\\a_1&x_1&m_1\\a_2&x_2&m_2\end{vmatrix},$$

and

$$F_{MYC}=\frac{1}{2}\begin{vmatrix}1&1&1\\m_1&y_1&c_1\\m_2&y_2&c_2\end{vmatrix}.$$

The sides composing the equal angles are $d(M,A)=(a_1-m_1)$, $d(A,X)=(x_1-a_1)$, $d(Y,C)=(c_1-y_1)$, and $d(C,M)=(m_1-c_1)$. For the directed angles $\alpha$ and $\alpha'$ we have

$$\alpha=\angle\left(\overrightarrow{MA},\overrightarrow{AX}\right)=\frac{x_2-a_2}{x_1-a_1}-\frac{a_2-m_2}{a_1-m_1}$$

$$\alpha'=\angle\left(\overrightarrow{YC},\overrightarrow{CM}\right)=\frac{m_2-c_2}{m_1-c_1}-\frac{c_2-y_2}{c_1-y_1}$$

$$\alpha=\alpha' \quad \Rightarrow \quad \frac{x_2-a_2}{x_1-a_1}-\frac{a_2-m_2}{a_1-m_1}=\frac{m_2-c_2}{m_1-c_1}-\frac{c_2-y_2}{c_1-y_1}$$

$$\Rightarrow \quad \frac{x_1m_2-x_2m_1-a_1m_2+a_2m_1+a_1x_2-a_2x_1}{y_1c_2-y_2c_1-m_1c_2+m_2c_1+m_1y_2-m_2y_1}=$$

$$=\frac{a_1x_1-x_1m_1+m_1a_1-a_1^2}{m_1c_1-m_1y_1+c_1y_1-c_1^2}.$$

The latter equation can be reach writing extensively equation (18). □

Let's apply now lemma 5 on the following pairs of allowable triangles:

$\triangle MAX$ and $\triangle YCM \Rightarrow$

$$\frac{F_{MAX}}{F_{YCM}}=\frac{d\left(M,A\right)\cdot d\left(A,X\right)}{d\left(Y,C\right)\cdot d\left(C,M\right)}, \qquad (19)$$

$\triangle CMY$ and $\triangle DMX \Rightarrow$

$$\frac{F_{CMY}}{F_{DMX}}=\frac{d\left(C,M\right)\cdot d\left(M,Y\right)}{d\left(D,M\right)\cdot d\left(M,X\right)}, \qquad (20)$$

$\triangle XDM$ and $\triangle MBY \Rightarrow$

$$\frac{F_{XDM}}{F_{MBY}}=\frac{d\left(X,D\right)\cdot d\left(D,M\right)}{d\left(M,B\right)\cdot d\left(B,Y\right)}, \qquad (21)$$

$\triangle YMB$ and $\triangle XMA \Rightarrow$

$$\frac{F_{YMB}}{F_{XMA}}=\frac{d\left(Y,M\right)\cdot d\left(M,B\right)}{d\left(X,M\right)\cdot d\left(M,A\right)}. \qquad (22)$$

$$(19)\cdot(20)\cdot(21)\cdot(22)=\frac{F_{MAX}}{F_{YCM}}\cdot\frac{F_{CMY}}{F_{DMX}}\cdot\frac{F_{XDM}}{F_{MBY}}\cdot\frac{F_{YMB}}{F_{XMA}}=1$$

$$\Rightarrow \frac{d\left(A,X\right)\cdot d\left(M,Y\right)}{d\left(Y,C\right)\cdot d\left(M,X\right)}\cdot\frac{d\left(X,D\right)\cdot d\left(Y,M\right)}{d\left(B,Y\right)\cdot d\left(X,M\right)}=1$$

$$\Rightarrow \frac{d\left(A,X\right)\cdot d\left(X,D\right)}{d\left(B,Y\right)\cdot d\left(Y,C\right)}=\frac{d\left(M,X\right)\cdot d\left(X,M\right)}{d\left(M,Y\right)\cdot d\left(Y,M\right)}. \qquad (23)$$

According lemma 3, and using the notation given in (4), we have

$$d\left(A,X\right)\cdot d\left(X,D\right)=d\left(P,X\right)\cdot d\left(X,Q\right)=|s|^2-|x|^2, \quad (24)$$

and

$$d\left(B,Y\right)\cdot d\left(Y,C\right)=d\left(P,Y\right)\cdot d\left(Y,Q\right)=|s|^2-|y|^2. \quad (25)$$

Inserting (24) and (25) in (23) we obtain

$$\frac{|s|^2-|x|^2}{|s|^2-|y|^2}=\frac{-|x|^2}{-|y|^2} \Rightarrow |x|^2=|y|^2 \Rightarrow |x|=\pm|y|,$$

and finally, as it has been shown before,

$$|x|=|y| \Rightarrow d\left(X,M\right)=d\left(M,Y\right).\square$$

**Proof 4**

Let $k$ be a parabolic circle in $I_2$, and let $M$ be the midpoint of the chord $\overrightarrow{PQ}$ of $k$. Let's choose the coordinate system as shown (in the affine model) in figure 10, i.e, the tangent on the circle $k$ parallel to the chord $\overrightarrow{PQ}$ as the $x$-axis, and the isotropic straight line through $M$ as the $y$-axis.

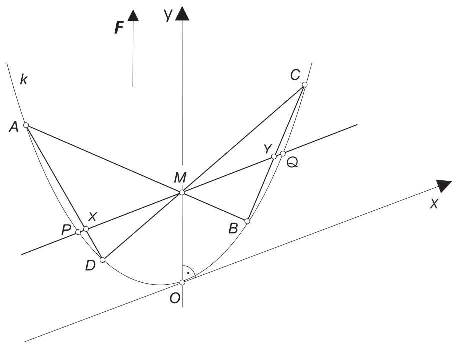


Fig. 10

Let $A\left(a_1,Ra_1^2\right)$, $B\left(b_1,Rb_1^2\right)$,$A\neq B\Rightarrow a_1\neq b_1$, and $C\left(c_1,Rc_1^2\right)$, $D\left(d_1,Rd_1^2\right)$, $C\neq D\Rightarrow c_1\neq d_1$, be four points on the parabolic circle $k$. Choosing $M\left(0,m\right)$, for the chord $\overrightarrow{PQ}$ we have $\overrightarrow{PQ}\equiv y=m$. Besides, for $\overrightarrow{AB}$ being a chord through $M$, the following relations are obtained:

$M$, $A$, $B$ collinear points $\Leftrightarrow$

$$\begin{vmatrix} 0 & m & 1 \\ a_1 & Ra_1^2 & 1 \\ b_1 & Rb_1^2 & 1 \end{vmatrix}=0\Leftrightarrow a_1b_1=-\frac{m}{R}. \qquad (26)$$

Analogously, for $\overrightarrow{CD}$ being a chord through $M$, we have:

$M$, $C$, $D$ collinear points $\Leftrightarrow$

$$\begin{vmatrix} 0 & m & 1 \\ c_1 & Rc_1^2 & 1 \\ d_1 & Rd_1^2 & 1 \end{vmatrix}=0\Leftrightarrow c_1d_1=-\frac{m}{R}. \qquad (27)$$

Let's denote further on $X\left(x_1,m\right)$ and $Y\left(y_1,m\right)$.

One obtains the following:

$A$, $D$, $X$ collinear points $\Leftrightarrow$

$$\begin{vmatrix} x_1 & m & 1 \\ a_1 & Ra_1^2 & 1 \\ d_1 & Rd_1^2 & 1 \end{vmatrix}=0\Leftrightarrow Rx_1\left(a_1+d_1\right)=m+Ra_1d_1. \qquad (28)$$

$C$, $B$, $Y$ collinear points $\Leftrightarrow$

$$\begin{vmatrix} y_1 & m & 1 \\ b_1 & Rb_1^2 & 1 \\ c_1 & Rc_1^2 & 1 \end{vmatrix}=0\Leftrightarrow Ry_1\left(b_1+c_1\right)=m+Rb_1c_1. \qquad (29)$$

Finally, using (26), (27), (28), and (29) it follows:

$$x_1+y_1=\frac{m+Ra_1d_1}{R(a_1+d_1)}+\frac{m+Rb_1c_1}{R(b_1+c_1)}=$$

$$=\frac{(m+Ra_1d_1)(b_1+c_1)+(m+Rb_1c_1)(a_1+d_1)}{R(a_1+d_1)(b_1+c_1)}=$$

$$=\frac{R(a_1b_1d_1+a_1c_1d_1+a_1b_1c_1+b_1c_1d_1)+m(a_1+b_1+c_1+d_1)}{R(a_1+d_1)(b_1+c_1)}=$$

$$=\frac{R\left(-\frac{m}{R}d_1-\frac{m}{R}a_1-\frac{m}{R}c_1-\frac{m}{R}b_1\right)+m(a_1+b_1+c_1+d_1)}{R(a_1+d_1)(b_1+c_1)}=0$$

$\Rightarrow M$ is the midpoint of $\overrightarrow{XY}$. □

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