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ABSTRACTS

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Contents

Plenary lectures

Of Peacock Feathers, Ellipses on Toruses, Crop Circles and other Mysterious Things

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Geometry is one of the oldest sciences in human history. With the help of fast computers, its applications have gone beyond our wildest dreams of a few decades ago. In this talk, the authors want to show their way of tackling some exciting geometric "daily-life applications". The results are useful for explaining more complicated considerations, and in some cases, they might even lead to otherwise hard to achieve new results.

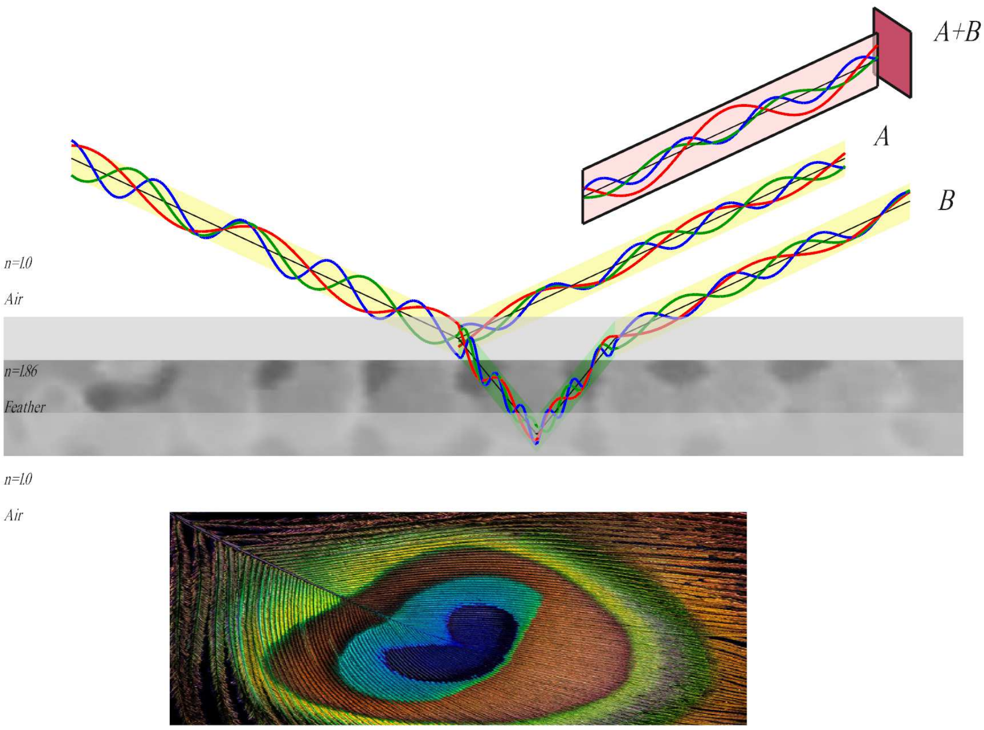

Figure 1: The different colors of a peacock feather are not due to pigments, but are rather the result of multiple refractions, multiple total reflections and eventually interferences of light waves. For accurate simulation, the spectral colors have to be treated separately and after that combined into the final color.

How do the colors of peacock feathers come about (Fig.1)? It is known that they are not derived from pigments, but which conditions lead to which result? For better understanding, we have developed tools to simulate the propagation of light waves after reflection, refraction and diffraction at the double slit.
The residual figures give three more examples of the contents of the talk (the accompanying text is self-explanatory).

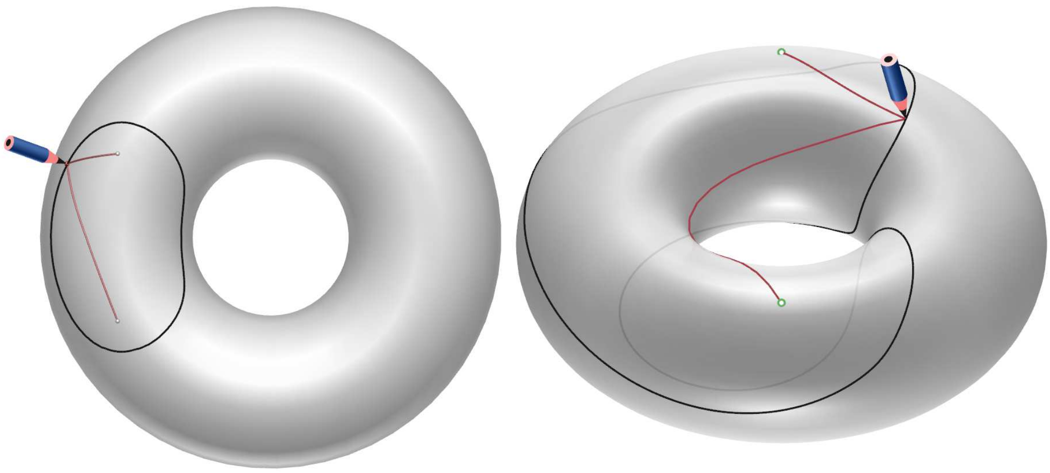

Figure 2: "Ellipses on a torus": If we consider an ellipse as the locus of all points with a constant sum of the distances to two fixed points, then we can also define such curves on doubly curved surfaces like a torus. The results can look quite different.

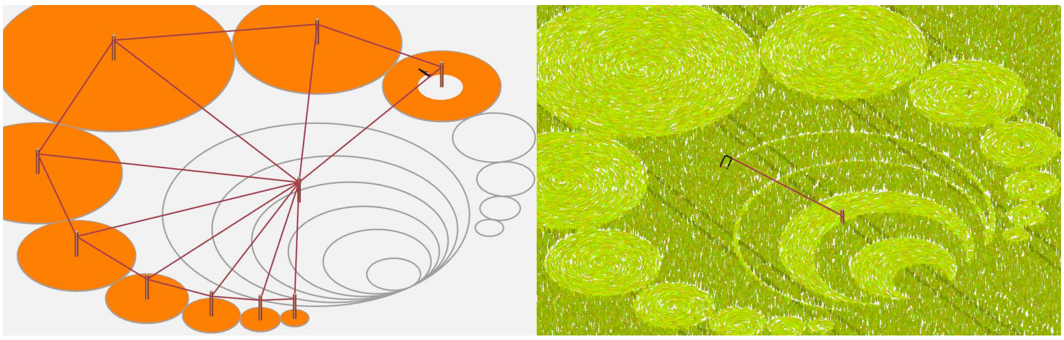

Figure 3: A non-negligible number of people still believes that some crop circles have be made by aliens in order to communicate with humans. However, it is possible for a small group of people to produce such geometric patterns within a few nocturnal hours. We present an example of how to proceed. In many cases, the always existent tracks of agricultural machines play an important role or are at least helpful.

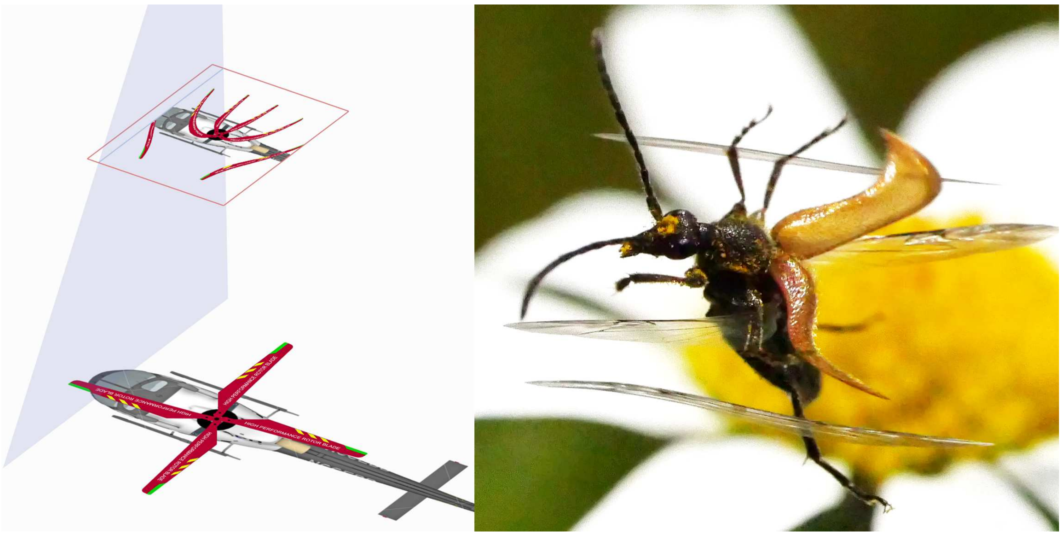

Figure 4: Most people believe that photographic images depict reality. This is, however, not always the case. The so-called rolling shutter effect may appear quite heavily when we deal with high speed motions. It does not help when we choose an extremely short aperture time: The image is stored "line by line", and this procedure often takes much longer than the aperture time. During that short span of time, e. g., wings of insects may move to completely other positions, appearing completely distorted but still sharp in the image.

Key words: light propagation, non-Euclidean geometries, crop circles, rolling shutter effect

MSC 2010: 51P05, 51K05, 51M09

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Representation and Design of Ruled Surfaces Based on the Dual Unit Sphere

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The development of concrete shell structures based on ruled surfaces was a project within the Priority Programme "Concrete Light" (SPP 1542) of the German Research Association (DFG). Within the project, the four-dimensional manifold of oriented lines (spears) in the three-dimensional Euclidean space was handled with the help of the dual unit sphere [1]. One of the outcomes was the add-on LineGeometry for the plug-in Grasshopper of the CAD-software Rhinoceros®.

This talk will present visualizations on the dual unit sphere (Fig. 1), strategies and examples for the design of ruled surfaces from control lines, and applications for the add-on LineGeometry. Furthermore, a short summary of the summer school "Line Geometry for Lightweight Structures (LGLS)" will be given.

Key words: ruled surfaces, dual unit sphere, dynamic relaxation, line geometry, form finding

MSC 2010: 53A25, 51M30, 70G65

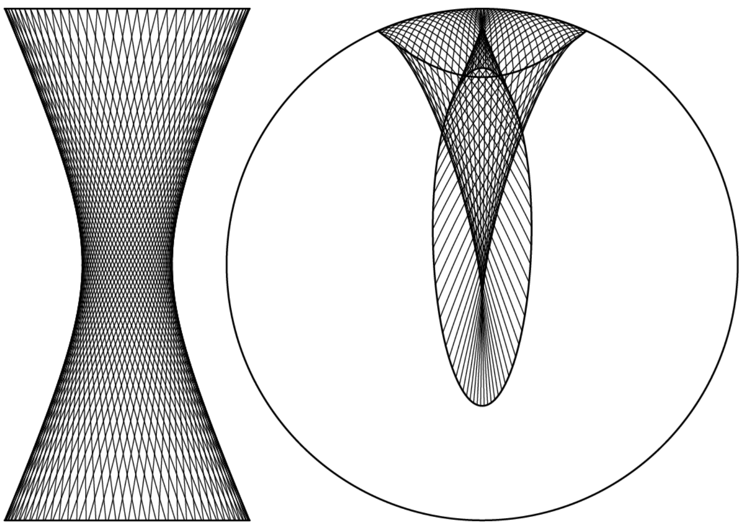

Figure 1: Hyperboloid of one sheet and its representation based on the dual unit sphere

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Geometric Genesis and Form Variation of Focal-Directorial Curves and Surface

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In this work, the starting point was a well-known geometric problem stated by a French mathematician Pierre de Fermat in the 17^{th} century. The problem is as follows: in Latin "datis tribus punctis, quartum reperire, a quo si ducantur tres rectae ad data puncta, summa trium harum rectarum sit minima quantitas" or in the English translation "for three given points, the fourth is to be found, from which if three straight lines are drawn to the given points, the sum of the three lengths is minimum", [3], [7].

The geometric solution to this problem was given by Evangelista Torricelli, who showed that a point satisfies the demands, [9]. This point, called the Fermat-Torricelli point, is considered to be the fifth significant point of a triangle. Furthermore, the Fermat-Torricelli point can be determined as a minimal value of the parameter S in the expression $R_1 + R_2 + R_3 = S$ which generates a trifocal curve. This curve was introduced by James Clerk Maxwell [5], [7]. In the 20^{th} century while solving some optimization problems Alfred Weber, an Austrian economist, generalized the Maxwell's expression introducing weight coefficients so that the expression becomes $a_1R_1 + a_2R_2 + a_3R_3 = S$. Curves obtained through this expression are known as isocost curves, [10], [7].

Introducing the duality in plane between points and straight lines we can define the following problem: for three given triangle sides (straight lines), the point is to be found, from which if three normals are drawn to the given sides, the sum of the three normals' length (distances) is minimum. The three-directrix curve can be expressed by $r_1 + r_2 + r_3 = S$. This curve, generalized by introducing weight coefficients, i.e. $b_1r_1 + b_2r_2 + b_3r_3 = S$, was defined for the first time in [1].

Combining these two problems, a correspondence with some geometric inequalities of polygons can be noticed. This fact was of use to the definition of transitory type of plane curves which we name Weberian focal-directorial curves (WFDC), [1]. The generalization comprises the introduction of an arbitrary number of foci and directrices, so that the following definition $a_1R_1 + ... + a_mR_m + b_1r_1 + ... + b_nr_n = S$ was obtained. Herewith, we consider particular variations of the WFDC with respect to the starting parameters: disposition of foci/directrices, change of weight coefficients $a_1, ..., a_m$; $b_1, ..., b_n$, and sum of distances variation; see Fig. 1 and Fig. 2.

In 3D space, we generalize this problem to surfaces' generations by introducing directorial planes. Therefore, three geometrical elements: point as a focus, straight line and plane as directrices are the directors for Weberian focal-directorial surfaces (WFDS), being geometric locus in space of the constant sum of distances to m foci,

n line directrices and k plane directrices defined by the following expression $a_1R_1 + ... + a_mR_m + b_1r_1 + ... + b_nr_n + c_1h_1 + ... + c_kh_k = S$, [1]. Herewith, we consider particular variations of the WFDS with respect to the starting parameters: disposition of foci/directrices, change of weight coefficients $a_1, ..., a_m$; $b_1, ..., b_n$; $c_1, ..., c_k$, and sum of distances variation, as well. In Fig. 3, we present the variation of form of WFDS with six foci and six line directrices which coincide with vertices and basis edges and sides of hexagonal prism ($m = n = k = 6$) when weight coefficients are predefined, i.e. $a_1 = ... = a_m = 1$; $b_1 = ... = b_n = B_i < 0$, $i = 1, ..., 4$; $c_1 = ... = c_k = 0$; $S = 0$.

Geometric genesis of focal-directorial curves and surfaces turns out to be a suitable base for space design in architecture and urbanism as well as in other engineering fields applications. Some proposals in cupolae design can be found in [2].

Key words: Fermat-Torricelli point, focal curve and surface, directorial curve and surface, focal-directorial curve and surface

MSC 2010: 51N20, 53A04, 53A05

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$m = n = 3;$ $m = n = 4;$ $m = n = 5;$ $m = n = 6;$

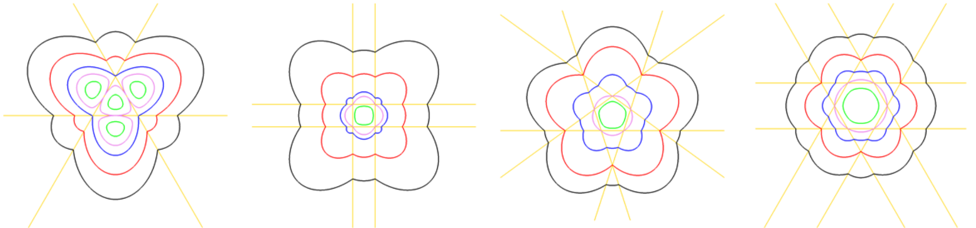


Figure 1: Variations of WFDC with parameters: $a_1 = ... = a_m = 1$; $b_1 = ... = b_n = -1$; $S_i > 0$, $i = 1, ..., 5$

$m = n = 3;$ $m = n = 4;$ $m = n = 5;$ $m = n = 6;$

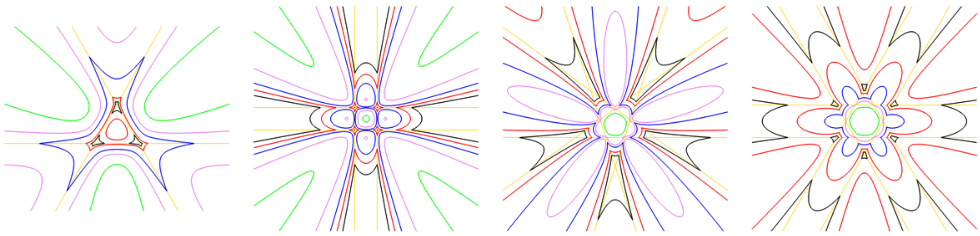
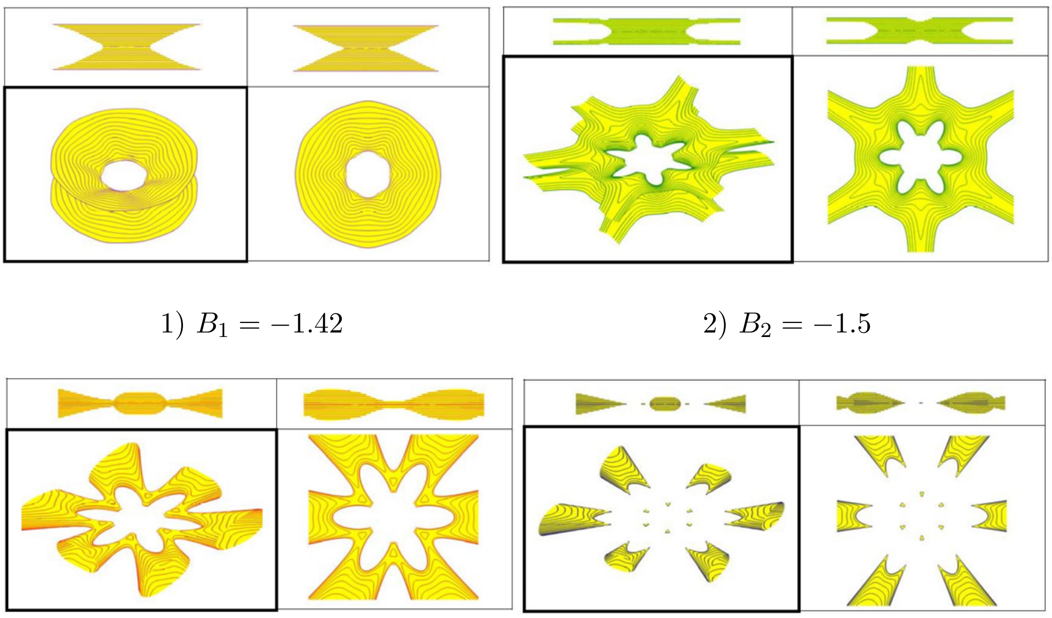


Figure 2: Variations of WFDC with parameters: $a_1 = ... = a_m = 1$; $b_1 = ... = b_n = B_i < 0$, $i = 1, ..., 5$; $S = 0$



1) $B_1 = -1.42$ 2) $B_2 = -1.5$

3) $B_3 = -1.55$ 4) $B_4 = -1.6$

Figure 3: Variations of WFDS

Architectural Geometry – Practical Use and Applications

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The entire field of architecture offers a broad field for geometric applications. It almost seems as if architecture has replaced kinematics as the "paradise" for the geometrician. An architectural project is always permeated by geometric questions from early design to production. Geometry is always confronted with different questions of different quality, especially when it comes to non-standard architecture. These questions are as follows:

- What is the right means of representation for the first design ideas? Should it be paper and pencil, a scaled model or already a CAD-package?
- While architectural design is always three-dimensional, plans and blueprints are often drawn two-dimensional. Why is there still such a gap between two- and three-dimensional information?
- An architect must always have an eye on the material properties. How can we enrich the theoretical, geometric model with material and structural information?
- A contemporary architect has to deal also with the manufacturing process, especially more and more with CNC fabrication and its programming. Is the architect of the future not only a designer but also a multi-trained expert between draughtsman, craftsman, programmer, IT- and geometry-expert?
- The robot-controlled construction site is a small dream, but one has to deal with it today. Where will architecture and the associated geometry go in the next 20 years?

Using examples from our courses, we present our approach and completed projects, as well as the geometric problems we have faced throughout the process, from design to production.

Contributed talks

Packing Stars in Fullerenes and other Polyhedra

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Let G and H be two simple connected graphs. An H**-packing** of G (or a packing of H in G) is a collection of vertex-disjoint subgraphs of G such that each component is isomorphic to H. If a packing is a spanning subgraph of G, we say that the packing is **perfect**.

A $(k, 6)$-fullerene graph is a planar, 3-regular and 3-connected graph with k-gonal and hexagonal faces. For $k = 5$ we have ordinary fullerene graphs, while for $k = 3$ or 4 we speak of generalized fullerenes.

It this contribution, we investigate the existence and properties of perfect packings of $K_{1,3}$ (and also some other small graphs) in ordinary and generalized fullerenes.

On Killing Magnetic Curves in $SL(2,\mathbb{R})$ Geometry

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Magnetic curves represent trajectories of charged particles moving on a Riemannian manifold under the action of a magnetic field.

A vector field X is a *Killing vector field* if the Lie derivative with respect to X of the ambient space metric g vanishes. The Killing vector field can be interpreted as an infinitesimal generator of isometry on the manifold in the sense that the flow generated by this field is a continuous isometry of the manifold.

The trajectories corresponding to the Killing magnetic fields are called the *Killing magnetic curves*. Killing magnetic curves in Minkowski space, Euclidean space, Sol space and $\mathbb{S}^2 \times \mathbb{R}$ space were studied in [1, 2, 3, 4], respectively.

We consider the Killing magnetic curves in $SL(2,R)$ space.

Key words: $SL(2,R)$ geometry, Killing vector field, magnetic curve

MSC 2010: 53C30, 53B50, 53Z05.

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Using the GWS Model in the Conceptual Development of Rotation

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The mathematical rigid transformation of rotation takes an important place in the education of geologist where structural lines and planes have often been rotated from some initial spatial orientation. Therefore, mathematical education of geologist is traditionally full of the use of instant constructive methods that need to be constantly adapted to the use of new artefacts. When using these methods, students often prefer to use previously memorized (often inaccurate) procedures rather than creating links within a given problem or drawing conclusions from a given representation.

This presentation reviews some psychological and educational research to highlight the problems in connection to the conceptual understanding of transformation of rotation. The author's teaching experience based on the use of *the Geometric Working Space model* is given. The GWS model enables understanding of the circulation of knowledge within a specific geometric task.

Key words: transformation geometry, conceptual knowledge, task analysis

MSC 2010: 97C30, 97C70, 97B40

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Dense Regular Ball Packings in Higher Dimensional Hyperbolic Spaces

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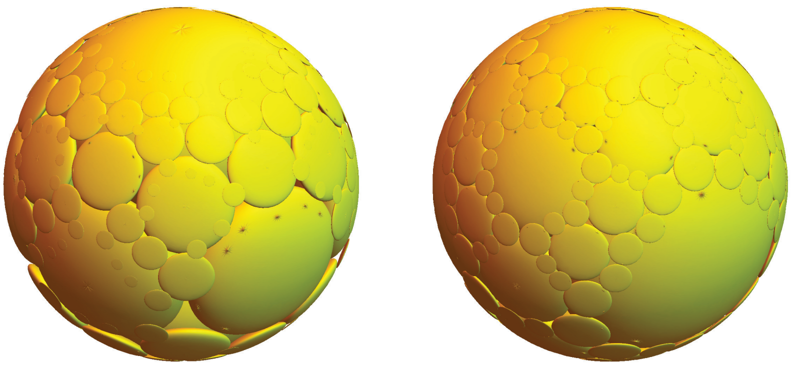
We discuss regular ball packings of hyperbolic space, where the symmetries of the packings are given Coxeter simplex groups. We produce the densest known horoball packings in dimensions $3 \leq n \leq 9$. We highlight a class of 3-dimensional packings that achieve the packing density the upper bound due to K. Böröczky, but with a different symmetry group, and provide visualizations in the projective model of hyperbolic geometry. Other packings we found include ones that exceed the conjectured 4-dimensional packing density upper bound due to L. Fejes-Tóth (Regular Figures, 1964) with densities of $\frac{5\sqrt{2}}{\pi^2} \approx 0.71644896$, and we mention the densest known packings in the remaining dimensions.
This is joint work with J. Szirmai.



The 5-dimensional Regular Solids Move on the Computer 2-screen with Visibility

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In previous works (see [1], [2], [3]) the authors extended the method of central projection to higher dimensions, namely, for $E^4 \to E^2$ projection from a one dimensional centre figure, together with a natural visibility algorithm. All these are presented in the linear algebraic machinery of real projective sphere PS^4 or space $P^4(\mathrm{V}^5, V_5, \sim)$ over a real vector space V^5 for points and its dual V_5 for hyperplanes up to the usual equivalence $\sim$ (expressed by multiplication by positive real numbers or non-zeros, respectively).

In this presentation we further develop this method for $E^5 \to E^2$ projection by the exterior (Grassmann - Clifford) algebra (with scalar product) and implement on computer with other effects of illumination, e.g. for (regular and other) polytopes on the base of the homepage `http://www.math.bme.hu/~prok`. The machinery is applicable for any d-dimensional projective space P^d and p-dimensional image.

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Remarks on Algebraic Geodesics on Quadrics

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Geodesics on quadrics, especially on ellipsoids, have frequently attracted the interest of mathematicians. It is well-known that (non-algebraic) geodesics on ellipsoids oscillate between lines of curvature. Further, the tangents of a geodesic on a quadric $\mathcal{Q}$ are tangents to a confocal quadric $\mathcal{F}$. Therefore, the tangent developable of a geodesic on $\mathcal{Q}$ is a developable ruled surface in the congruence of common surface tangents of $\mathcal{Q}$ and $\mathcal{F}$.

Y.N. Fedorov has shown in [1] how to construct algebraic geodesics on quadrics. Based on the complicated, but efficient construction, A.M. Perelomov gave a few examples of closed, algebraic geodesics on all affine types of quadrics in [4, 5].

In this presentation, we add some more results on low degree algebraic geodesics on quadrics. Besides the obvious existence of rational parametrizations in the very low degree cases, we find that algebraic geodesics on quadrics can be quartic space curves of the first and second kind as well. On paraboloids, even cubic space curves can occur as algebraic geodesics.

Key words: geodesic, algebraic curve, closed curve, quadric

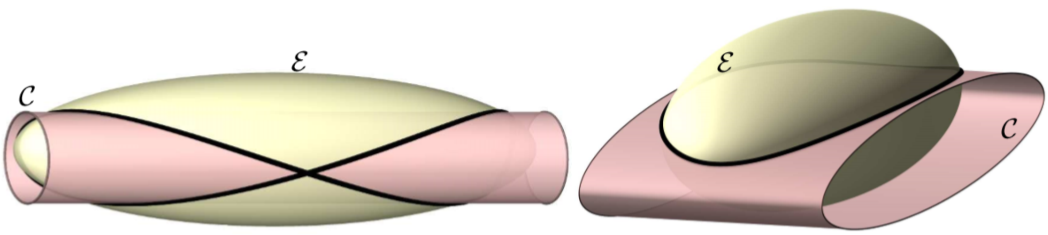

Figure 1: Closed algebraic geodesics on a triaxial ellipsoids: a geodesic with a double point (left), without double point (right). In both cases we have chosen the shape parameter $a = 4$.

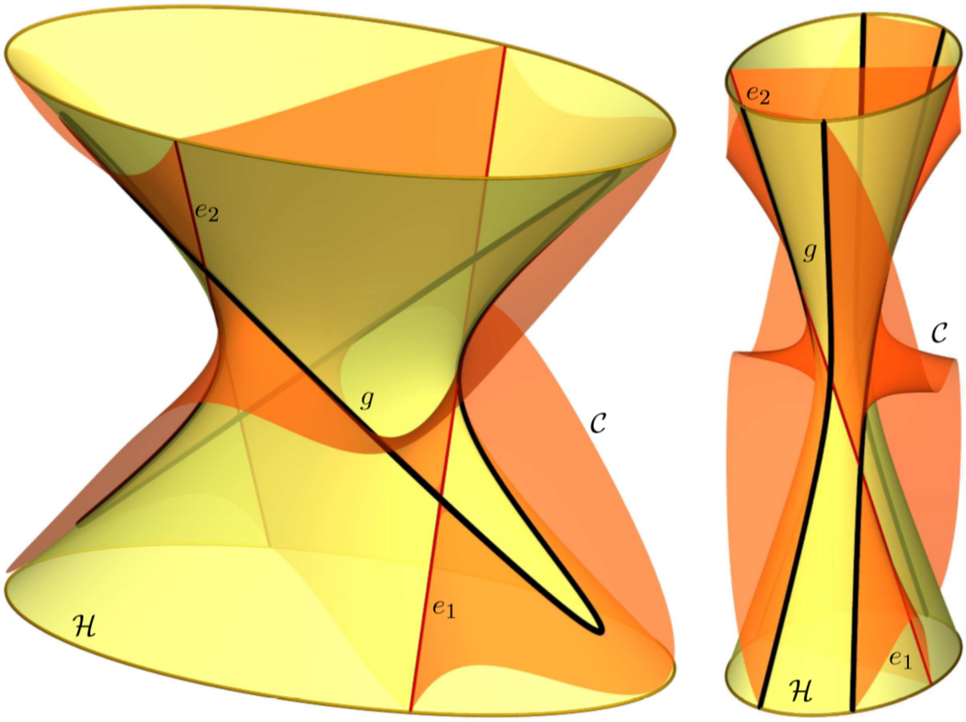

Figure 2: Left: The rational geodesic g on the one-sheeted hyperboloid $\mathcal{H}$ is a quartic of the second kind. The curve g is a part of the intersection of the hyperboloid $\mathcal{H}$ and a cubic surface $\mathcal{C}$. The two rulings e_1 and e_2 are also common to both, $\mathcal{H}$ and $\mathcal{C}$. Right: Another geodesic of the same type shows that these curves can have real ideal points.

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The Neighborhood Complexes of Almost s-stable Kneser Graphs

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In 1978, László Lovász proved the famous Kneser Conjecture concerning the chromatic number of the Kneser graphs $KG_{m,n}$ by introducing the neighborhood complex. In the same year, Alexander Schrijver defined certain induced subgraphs of $KG_{m,n}$ – called the stable Kneser graphs $SG_{m,n}$ – and showed that they are vertex-critical. Schrijver used another, Bárány's method, to obtain the chromatic number of the stable Kneser graphs. Almost 25 years later, in 2002, Anders Björner and Mark de Longueville studied the neighborhood complex of $SG_{m,n}$, and determined its homotopy type [1]. Frédéric Meunier generalized Schrijver's construction and formulated the conjecture on the chromatic number of the s-stable and almost s-stable Kneser graphs. We determined the homotopy type of the neighborhood complex of the almost s-stable Kneser graphs [2]. In conjunction with Lovász's topological bound on the chromatic number, we gave the chromatic number of these graphs, which was recently determined by Chen using other methods [3].

Key words: almost stable Kneser graph, neighborhood complex, homotopy type

MSC 2010: 55P15, 05C15

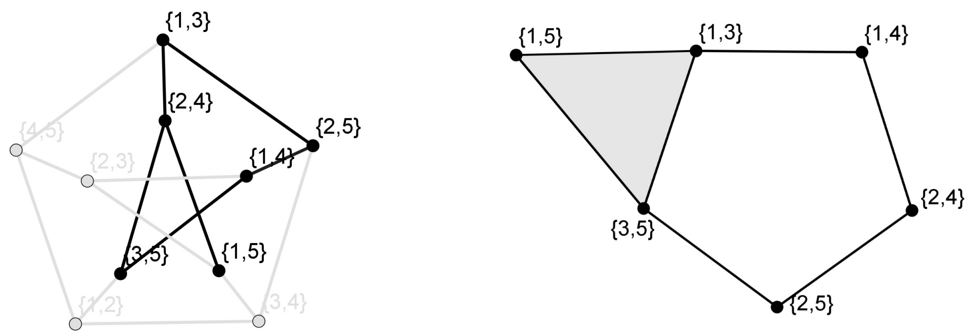

Figure 1: The almost 2-stable Kneser graph $SG^{\sim}_{5,2}$ and its neighborhood complex of it.

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Playing Architecture via Geometry and Graphics: Testing Unreal Engine

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The process of digital and technological evolution in recent years has shifted to words such as Virtual Reality and Artificial Intelligence. However, the use of these and other technologies in architecture is still limited by a lack of IT tools, which have only been made fully available in the last few years. The evolution of architectural design tools can be condensed into a sequence: (analogue) Drawing Board, AutoCAD, Game Engine, also representative of the historical contexts in which they have been developed and used. This study, based on a Master thesis recently discussed at the Politecnico di Milano, examines the role that *Game Engines* can play in the graphic representation and design processes. More specifically, it takes a closer look at the *Unreal Engine* release 4 (UE4) as a tool for creating a real-time design environment and for using Artificial Intelligence (AI) technologies to represent users' flows in the space, which can be adopted to carry out design strategies and to evaluate design options. For this purpose, various simulations have been developed, either considering the pedestrian planar flows interlinked with the form of space, either parametrically generating the spaces on the bases of the number of users. The first series of tests is then based on pre-assigned spatial contexts. In order to test the AI programmed sets, different situations were figured out and modeled in advance. Given the assigned space, a series of points has been subsequently assigned, working as 'attractors' according to possible users' interest locations, and a virtual robot (silhouette) has been placed to explore the various possible paths, based on a random sequences of movements towards the assigned attractor-points. In order to be able to graphically represent the visual simulations, the silhouette has been equipped with a tracing video-camera system shoving at the same time its movements and the scene from the camera point of view, and allowing to reproduce in real-time the flows as graphic diagrams in the space. The AI system was also tested in a 3D spatial context characterized by differences in heights, such as inclined corridors, vertical lifts, and so on. A second series of tests has been carried out considering the inverse process, that is, implementing a generative system able to create new spaces, such as rooms and paths, according to the needs emerged from the real-time analysis of the parametrically assigned users' flows. Therefore, a generative algorithm was set, able to update the geometry of space according to the number of people supposed to 'need space': in other words, space expanded according to the number of users. This generative process is based on a preliminary evaluation of the entire generative process, which essentially controls the generation itself in order to match 'rooms and paths' with the number of users. Further developments can of course explore more

advanced aspects, like patterns referring to perceptual senses (sight, hearing, or tactile, also including external events, and so on), and translate them into the virtual environment, based on similar parametric operations. Another extremely important issue is the psychological behavioral factor linked to the movement of the individuals in relation to the masses and vice versa. This last point introduces another relevant subject linked to the 'realism' of the context of the AI environment. In this case, given the difficulty of translating behavioral psychological aspects into appropriate descriptive codes because of their probabilistic nature, it would be appropriate to introduce neural networks technologies based on machine learning and deep learning systems, what we are aiming to do in the future. What we developed here with *Unreal Engine* would only shows the power and the potentiality of this typology of software, which is still to be fully discovered, since it has been available only few years ago, and the software houses are only recently getting increasingly interested in the architectural field. However, considering the present state of art, we tried to propose some little tests on if and how it is possible to use the system outside of its native target environment, adapting it to an architectural design spatial context. In our case the thesis, which was at the origin of this work, focused on the use of UE4 to realize an AI system helpful to represent and control – visually and parametrically – pedestrian flows in a three-dimensional environment, either pre-existing or generated according to specific inputs. More generally, linking analysis and project, especially in more complex scenarios, it can serve as a tool for mapping and analyzing architectural contexts, as well as for implementing, verifying, and comparing design choices, that is, efficiently sustaining the whole chain of the architectural design process.

Key words: artificial intelligence, flows, parametric modeling, dynamic environment, Game Engines, simulations

MSC 2010: 00A66, 51N05, 01A05, 97U99

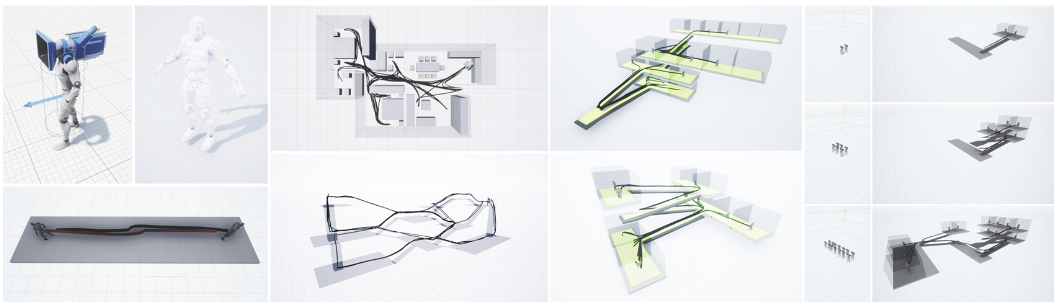

Figure 1: Synthesis and results of the process implemented for the tests (models by authors).

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Focal Sets of B-scrolls in Lorentz-Minkowski 3-space

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In this presentation we analyse focal sets of B-scrolls in Lorentz-Minkowski space. B-scrolls are examples of Lorentzian surfaces having no Euclidean counterparts, characterized by the property that their shape operator is not diagonalizable ([1], [2]). They are all ruled surfaces having null (lightlike) rulings. In [3] it was shown that their focal set degenerates to a curve which is of either null or spacelike causal character. In this work we analyse B-scrolls having a spacelike curve as the focal set.

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Orthogonality in $\mathbb{M}_n(\mathbb{C})$ and Geometry of the Numerical Range

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Let $\mathbb{C}^n$ be the linear space of all ordered n-tuples $x = (x_1, \ldots, x_n)$ of complex numbers equipped with the Euclidean norm $\|x\| = (\sum_{i=1}^n |x_i|^2)^{1/2}$. Let $\mathbb{M}_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices with the norm defined by $\|A\| = \max_{\|x\|=1} \|Ax\|$, $A \in \mathbb{M}_n(\mathbb{C})$. In this talk, we consider two types of orthogonality in $\mathbb{M}_n(\mathbb{C})$: the Birkhoff–James and the Roberts orthogonality. Some geometric properties of the (generalized) numerical range of a matrix are described in terms of these orthogonalities.

The talk is based on the results coming from joint works with Ljiljana Arambašić (University of Zagreb) and Tomislav Berić (University of Zagreb).

This research was supported by the Croatian Science Foundation under the project IP-2016-06-1046.

Key words: Birkhoff–James orthogonality, Roberts orthogonality, numerical range

MSC 2010: 47A12, 46B20, 47A30

Geometry of Split Quaternion Factorization

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The factorization of a quadratic split quaternion polynomial C has been investigated in [1] and, more recently, in [2]. The complete discussion requires a number of case distinctions, including inequality constraints and square roots. Moreover, the authors of both papers do not consider the case when the norm polynomial of C vanishes.

We present a complete *geometric* classification of factorizability of C in terms of the rational curve γ parameterized by C in the projective space of split quaternions. It is based on the type of γ (conic, line, point) and on its intersection points with the quadric N of non-invertible split quaternions. The main statements are:

- A factorization exists if γ is a conic (possibly on N), a null line, or just a point.
- If γ spans a non-null line L, a factorization exists if and only if L intersects N in at least one point and in the same number of points as the point set parameterized by C. (Only real intersection points are to be considered.)

Our geometric interpretation not only provides a rather simple geometric characterization of factorizability but also sheds new light on algorithms for computing factorizations, also over other quaternion algebras. Some aspect generalize to higher degree polynomials.

This is joint work with Daniel F. Scharler and Johannes Siegele and is supported by the Austrian Science Fund (FWF): P 31061 (The Algebra of Motions in 3-Space).

Key words: split quaternions, matrix polynomial, factorization

MSC 2010: 12D05, 51N15, 51N25

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Revisiting Quadrics: String Constructions and Movement of Conics

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The lecture consists of two parts: According to D. Hilbert, Staude's string construction of quadrics was one of the great mathematical results of the 19th century. We give a synthetic approach to these constructions, thus reducing the proof to existence theorems of differential equations.

The second part addresses a problem once posed by H. Brauner: There is a three-parametric set of planes cutting a given quadric $\mathcal{Q}$ along central conics. But the size of these conics depends only on its two semiaxes. Thus, there exist ellipses or hyperbolas with a one-parameter set of congruent copies on $\mathcal{Q}$. We present parametrizations for the movements of conics on ellipsoids and hyperboloids. There is a close connection to the theory of confocal quadrics.

Key words: string construction of quadrics, confocal quadrics, conics on quadrics

MSC 2010: 51N20, 53A17

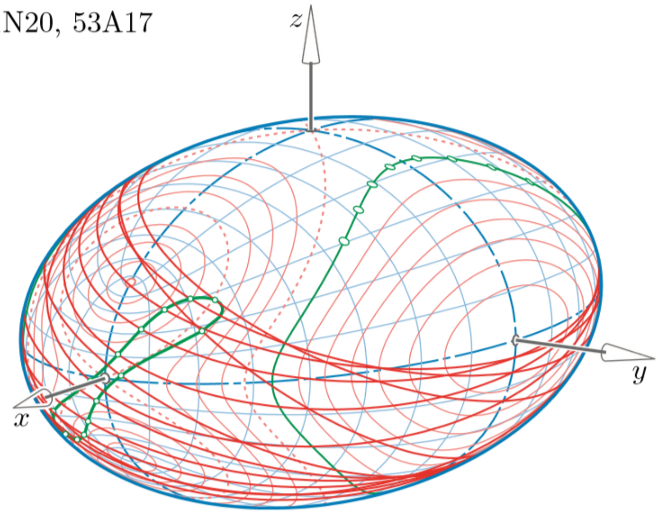

Figure 1: Movement of an ellipse on an ellipsoid with the trajectory of a vertex

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On Some Properties of Parabola in Isotropic Plane

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In this talk we are planning to show some of properties of parabola in the isotropic plane. First, we give the properties of parabola in the Euclidean plane listed in [1]. Some of the properties are valid also in isotropic plane and some of them have similar analogues in the isotropic plane. The differences appear mainly in two cases. The first is when we deal with the perpendicular lines since in the isotropic plane they are usually switched to isotropic lines, and for the points lying on them we can not measure distance, but span. The second case appears when it comes to the sine of an angle since in the isotropic plane the role of the sine of angle plays the angle itself.

In the other part of the talk we will turn the focus on the circles of curvature at points of parabola. In [3] the authors have studied the curvature of the focal conic in the isotropic plane where the form of the circle of curvature at its points is obtained. Hereby, we discuss several properties of such circles of curvature at the points of a parabola in the isotropic plane.

Key words: isotropic plane, parabola, circle of curvature

MSC 2010: 51N25

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Generalization of Schönhardt's Polyhedron in the d-dimensional Euclidean Space

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A triangulation of a d-dimensional polytope $P \subset \mathbb{R}^d$ is a collection of d-dimensional simplices with the properties that their union is P, their vertices are vertices of P, and any two simplices intersect in a face (possibly empty). In two dimensions, it is well-known that every planar polygon can be triangulated.

In three dimensions, the situation is different: There are 3-dimensional polyhedra which are nontriangulable. The best-known example of such a polyhedron is Schönhardt's polyhedron, which is a twisted triangular prism, found by Schönhardt [1] in 1928. In fact, there are infinitely many Schönhardt's polyhedra, since the base of the twisted prism can be any triangle, and the twist angle may vary in an interval that depends on the shape of the triangle.

Let S be a Schönhardt's polyhedron. It is simply connected (even starlike), and every face of S is a triangle. S has 6 vertices, that is the smallest number of vertices of a nontriangulable polyhedron. S contains no tetrahedron whose vertices form a subset of the vertices of the polyhedron - this is implied by the fact that every diagonal (a segment connecting two vertices which are not connected by an edge) of S lies outside S. The polyhedron S can be obtained from its convex hull by removing three tetrahedra whose vertices are vertices of S and they have pairwise disjoint interiors.

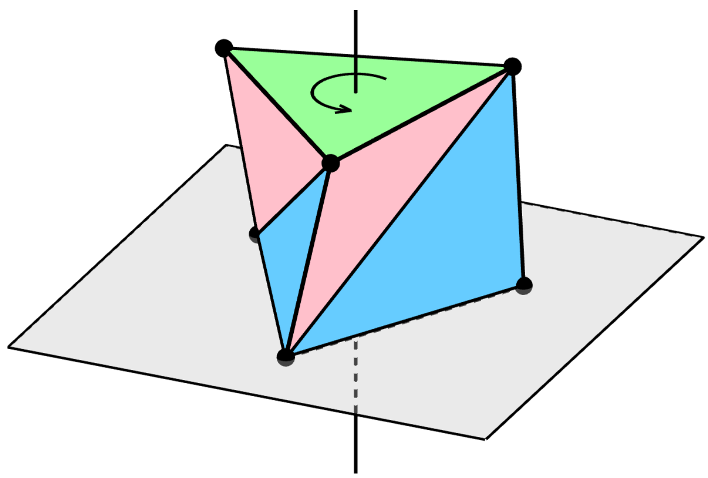

Figure 1: Schönhardt's polyhedron

We consider the problem how to generalize Schönhardt's polyhedron in the d-dimensional Euclidean space to get d-dimensional nontriangulable polytopes in $\mathbb{R}^d$, $d \geq 4$. We try to create polytopes with as many analogous properties to a Schönhardt's polyhedron as possible. We get some infinite families of such d-dimensional polytopes which are nontriangulable in a nontrivial way (a trivial way is, for example, when there is an at most $(d-1)$-dimensional face of the polytope which is nontriangulable). We also show that some analogous properties do not hold in any generalization of Schönhardt's polyhedron to higher dimensions.

Key words: polytope, triangulation, simplicial complex

MSC 2010: 51M20, 52B11, 52C22

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Beauties of Error Function

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The famous Gaussian integral, known also as the Euler-Poisson integral, of Gaussian function over the entire real axis was originally discovered in 1733 by Abraham de Moivre, while Gauss was the first to publish its precise form in 1809,

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

There are many applications of this integral in various fields. It is widely used in probability to compute the normalizing constant of the normal distribution and its cumulative distribution function. In physics, such integrals appear frequently in quantum mechanics, to find the probability density of the ground state of the harmonic oscillator and its partition function and propagator, and in statistical mechanics for the path integral formulation. This convergent improper integral can be geometrically interpreted as the area of infinite curvilinear trapezoidal region bounded from above by the standard normal distribution curve (Bell curve) and by coordinate x-axis from below. Value of this integral is closely related to the error function, which appears as the result of the integration commands in many computer algebra systems,

$$\int e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}\ \mathrm{erf}(x).$$

The error function is a special (non-elementary) function whose graph is of a sigmoid shape that occurs frequently in probability, statistics, and partial differential equations describing diffusion:

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt.$$

It can be also represented by a McLaurin series

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \cdots \right),$$

or as the continued fraction

$$\int_0^x e^{-t^2} dt = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) = \frac{\sqrt{\pi}}{2} - \cfrac{\frac{1}{2}e^{-x^2}}{x + \cfrac{1}{2x + \cfrac{2}{x + \cfrac{3}{2x + \cfrac{4}{x + \cdots}}}}}.$$

Some of the “beauties” of the Bell curve and the error function graph will be described and used in Minkowski point set operations to generate surfaces of unexpected forms and properties.

Key words: Error function, Gaussian integral, Minkowski combinations

MSC 2010: 33E20, 65D17

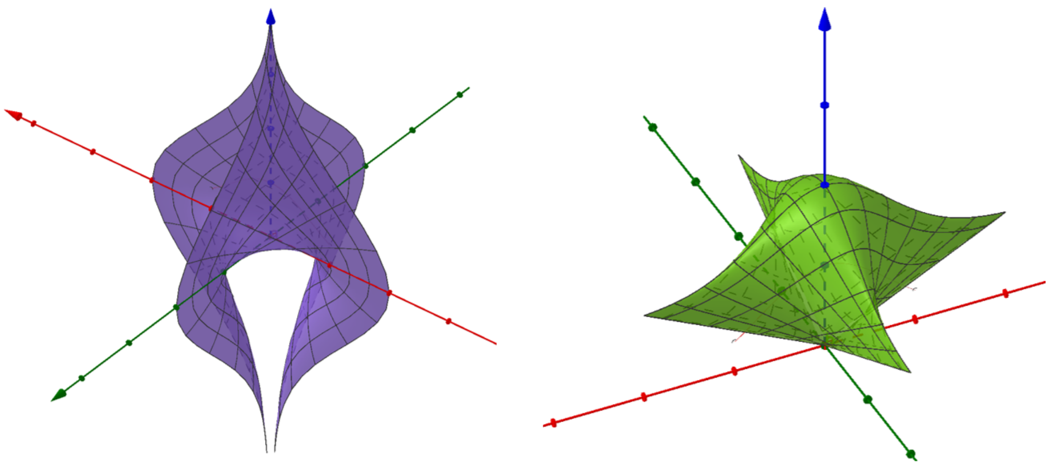

Figure 1: Minkowski sum and product of the normal distribution curves

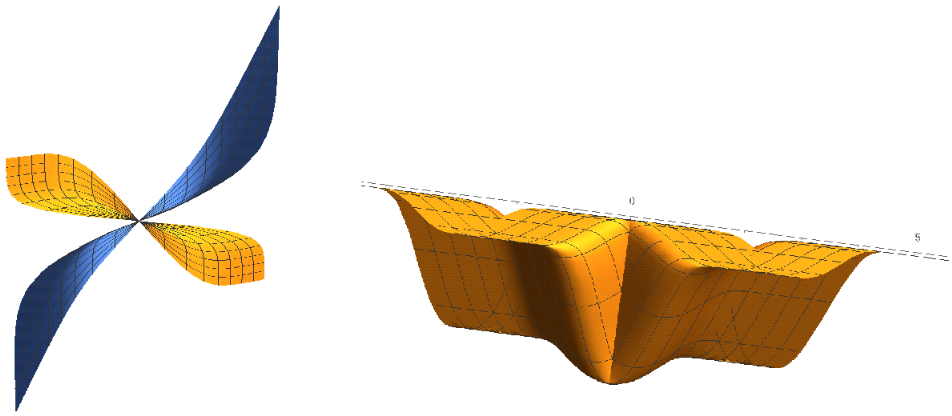

Figure 2: Minkowski sum and product of error functions

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On Parallelotope Configuration

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The parallelotope $\mathcal{P}$ is a convex polytope which fills the space facet to facet by its translation copies without intersecting by inner points. The centers of the parallelotopes form an n-dimensional lattice.

A plane configuration is a system of p points and l straight lines arranged in a plane in such a way that every point of the system coincides with straight lines of fixed number γ and every straight line of the system coincides with points of fixed number π.

In this paper I examine the connection between 3- and 4-dimensional parallelotopes and point-line configurations in the plane and space. Parallelotopes contain 2- or 3-belts therefore configurations have special forms. On the other hand we must generalize the concept of the configuration to be able to describe all parallelotopes. So I define the parallelotope configuration: p-configuration.

Key words: parallelotope, configuration

MSC 2010: 52B11, 51E20

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A Complete System of the Shapes of Triangles

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In this lecture, we will consider the shape of a triangle by means of a ternary operation which satisfies some properties. It will be proved that each system of the shapes of triangles can be obtained by means of the field with the defined ternary operation. A geometric model of the shapes of triangles on the set of complex numbers will be given as a motivation for introducing some geometric concepts. The concept of transfer will be also introduced and some properties of this concept will be investigated.

Key words: ternary operation, quasigroup, transfer

MSC 2010: 20N05

A Set of Planar and Spatial Tessellations Based on Compound 3D Models of the 8D and 9D Cubes

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The 3-dimensional framework (3-model) of any k-dimensional cube (k-cube) can be produced based on initial k edges arranged by rotational symmetry, whose Minkowski sum can be called zonotope. Combining $2 < j < k$ edges, 3-models of j-cubes can be built, as parts of a k-cube. The suitable combinations of these zonotope models can result in 3-dimensional space-filling mosaics. The investigated periodical tessellations always hold the 3-model of the k-cube and necessary j-cubes derived from it.

The mosaics can have fractal or fractal like structure as well, since the stones can be replaced with restructured ones. The hulls of 3-models of k- and $3 < j$-cubes can be filled with different sets of 3-models of $2 < j < (k-1)$ or $2 < i < (j-1)$ -cubes touching each other at congruent faces. Another possibility is if the 3-models of the given k- and of the derived j-cubes are arranged along the outer edges of the restructured models and the faces are replaced with central symmetrically arranged sets of the above elements. The inner space of the new compound models is filled also with the initial models. If also the inner edges are followed by 3-models of the k- and j-cubes, the construction can require unmanageable amount of the elements, from practical points of view, but the restructured mosaics can have a more consequent fractal like structure.

The intersections of the mosaics with planes allow unlimited possibilities to produce periodical symmetric plane-tiling. Moving intersection planes result in series of tessellations or grid-patterns transforming into each other. These can be shown in varied animations.

Key words: constructive geometry, hypercube modeling, tessellation, fractal

MSC 2010: 52B10; 52B12, 52B15, 65D17

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Angular Proximity between Conics & Quadrics

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The problem of touching of a pair of conics in the plane and a pair of quadrics in space, initially in arbitrary position, via rotation about a given fixed axis has not yet been treated in any available literature. Two approaches are discussed herein. In one case the entities are expressed in Cartesian coordinates, using symmetric matrices of coefficients and scalar equations. Constraint equations are developed in terms of shared point and tangent line or plane after rotation. In the other case, an entity is expressed in terms of parameterized points and the tangents are formulated as partial derivatives of these parameterized point coordinates. Fig. 1 shows the small ellipse rotated about the origin into four possible positions of tangency. Apart from the four real solutions shown the first approach produces a monomial in rotation half-angle tangent of degree 16. Will the second improve upon this?

Fig. 2 shows two ellipsoids in more or less arbitrary position. They do not touch, so a rotation to produce mutual tangencies is just as feasible as if they had been separated like the ellipses in Fig. 1, initially.

In Fig. 3 the blue and red horizontal sections show elliptical traces on planes containing the (marked) points of tangency. The monomial produced with the first approach is of degree 28 which is unsatisfactory because the parametric approach yields one of degree 24.

The small blue ellipse rotates clockwise to assume tangency with the larger blue ellipsoid section. Similarly, the small red ellipse rotates counter-clockwise until the marks coincide.

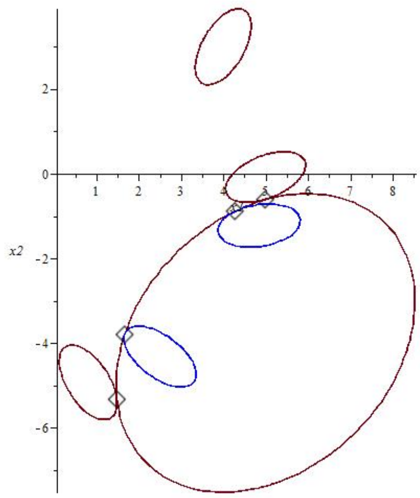

Figure 1: One fixed ellipse, the other in original and four tangential positions.

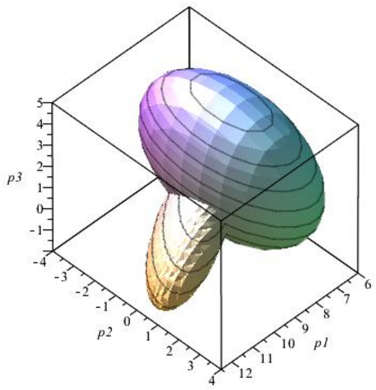

Figure 2: Two ellipsoids, intruding but not in tangency.

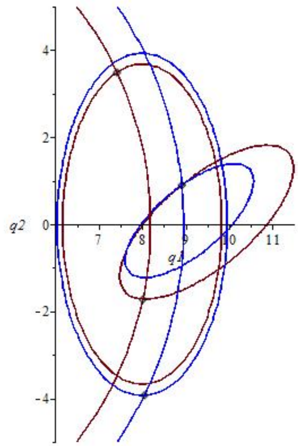

Figure 3: Ellipsoid horizontal sections in shared tangent planes.

Posters

On Certain Classes of Weingarten Surfaces in $SL(2, R)$ Space

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A Weingarten surface is a surface satisfying the Jacobi equation

$$\Phi(K, H) = \det \begin{pmatrix} K_u & K_v \\ H_u & H_v \end{pmatrix} = 0,$$

where K is the Gaussian curvature and H is the mean curvature of the surface.

The study of Weingarten surfaces was initiated by J. Weingarten in 1861. E. Beltrami and U. Dini few years later proved that the only non-developable Weingarten ruled surface in Euclidean 3-space is a helicoidal ruled surface. In the last decade several papers on Weingarten surfaces in different 3-dimensional spaces have appeared (see [1, 2, 3, 4]).

Motivated by the fact that there are no results about Weingarten surfaces in $SL(2, R)$ space, we examine some classes of Weingarten surfaces using the right half-space model of $SL(2, R)$ space.

Key words: $SL(2, R)$ geometry, Weingarten surface, ruled surface

MSC 2010: 53C30, 53A10.

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3D Graphic Statics via Grassmann Algebra

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The procedures for replacing a given force system with some other force system and procedures for finding equilibrating forces to the given force system described in the poster are carried out using geometric constructions (Figure 2) which can be considered as a partial three-dimensional extension of funicular polygon construction, [2], [4].
Geometrical constructions can be described in terms of basic incidence operations - meet and join, which we translated into Python code by using homogeneous coordinates of points, planes and also lines, where the latter is achieved by means of Grassmann algebra [1] in the extended Euclidean space.
Visualizations of the graphical procedures are made in Rhinoceros 3D using Grasshopper (Figure 1).

Key words: Grassmann algebra, Plücker line coordinates, 3D graphic statics, static equivalence, equilibrating forces

MSC 2010: 70C20, 68U05, 65D18

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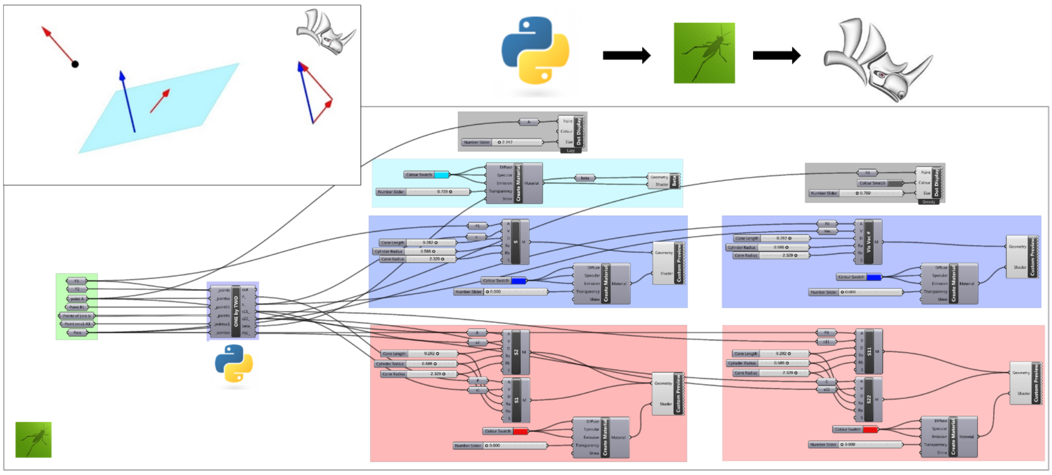

Figure 1: Program developed in Rhinoceros and Grasshopper for example for replacing single force with two forces, of which one acting at given point and the other lies in a given plane, which does not contain the point

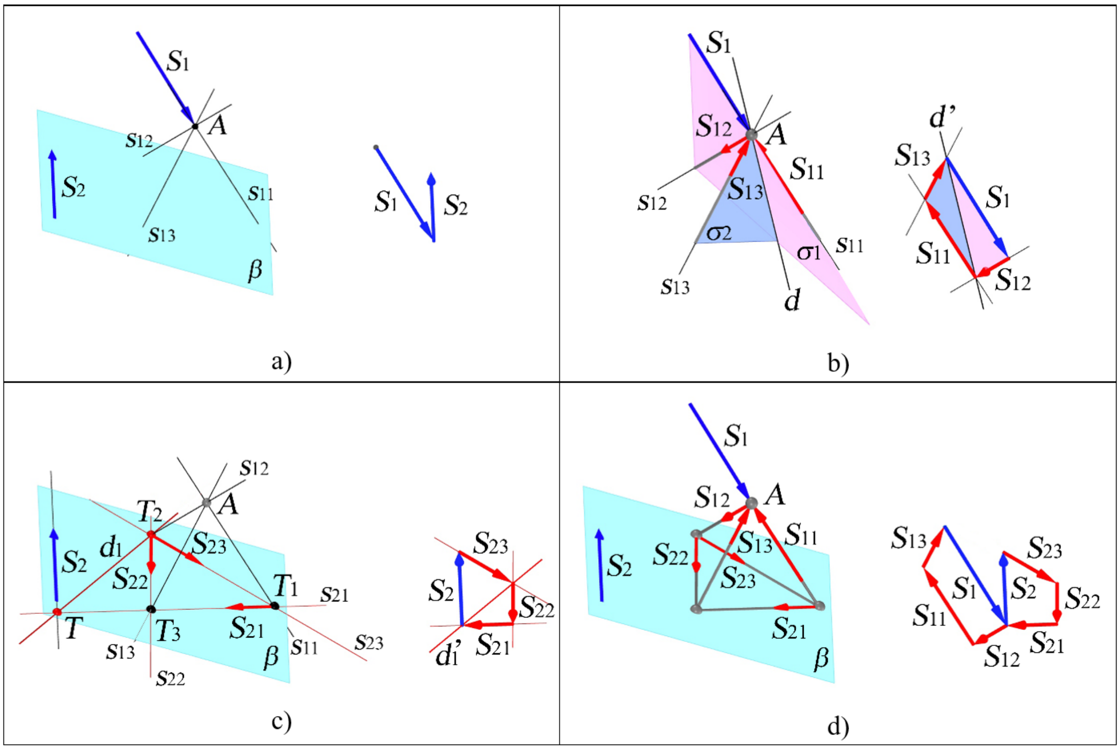

Figure 2: Procedure for finding equilibrating forces to the given two-force system along six given lines, that is along edges of given tetrahedron

An Artistic Approach to the Tesseract

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"Es ist die Freude an der Gestalt in einem hoheren Sinne, die den Geometer ausmacht." (Clebsch, in memory of Julius Plücker, Göttinger Abh. Bd. 15), [2].
This quote translated to English reads *"It is the joy in shapes in a higher sense, what makes a geometer"*, which is the perfect reflection of our feelings during our recent and ongoing mathematical/artistic project.
It started at the LGLS (Line Geometry for Lightweight Structures) Summer school in Dresden in October 2018 [4], and made to the art exhibition at the Bridges conference in Linz in July 2019. In this poster we show our artworks and the process of making it.

Figure 1: Detail from the exhibition in SLUB Dresden, photo by Masayo Aye

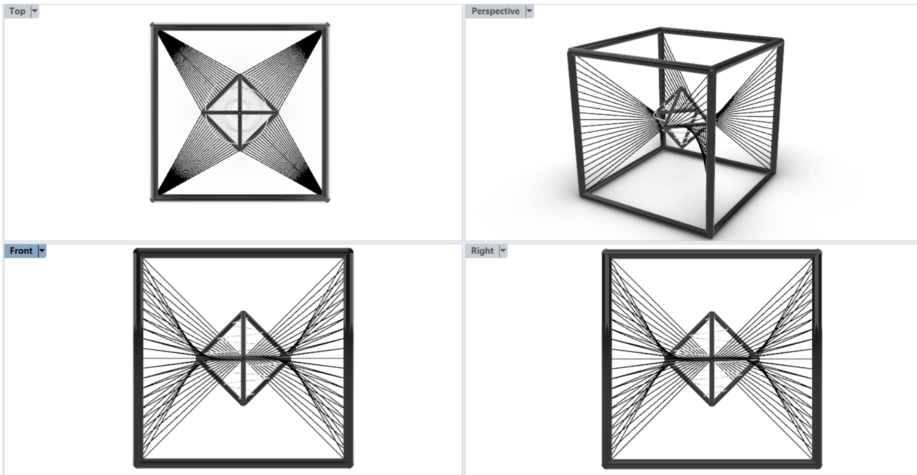

Figure 2: Digital model for one artistic portrayal of a tesseract

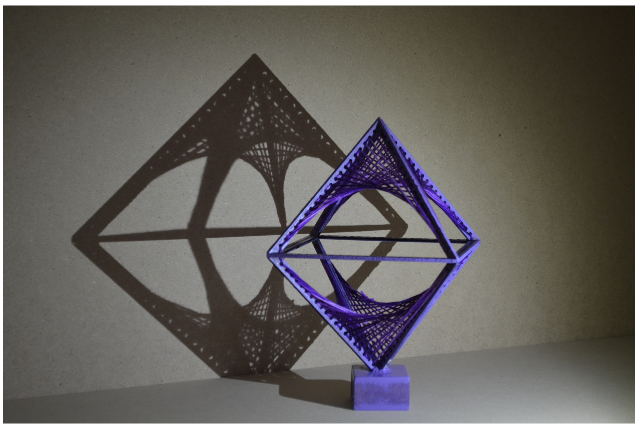

Figure 3: *Octahedron & 8 Parabolae*, [1]

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Parametrization of Null Scrolls as B-scrolls in 3-dimensional Lorentz-Minkowski Space

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In the Lorentz-Minkowski 3-space, every non-degenerate ruled surface with null rulings can be reparametrized as a B-scroll, i.e. as a ruled surface whose rulings correspond to the binormal vectors of a base curve. In this work we provide such a reparametrization by introducing a proper null frame for a base curve related to the rulings of a surface. By such approach, we can establish relations between lightlike curvature of a base curve and curvatures of a null scroll. It is also shown that every null curve lying on a null scroll can be used as a base curve for such reparametrization.

Key words: Lorentz-Minkowski space, null scroll, B-scroll

MSC 2010: 53A35, 53A55, 53B30, 53C50

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On Applications of Focal-Directorial Surfaces

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The innovations in creating complex geometric forms are in a high demand in various engineering fields, particularly in architecture, which tends to be a never ending source of ideas for shaping our living space both the interior and exterior. Therefore, the development of novel methods for generating new geometric forms became an everlasting topic.

We investigate the possibilities for obtaining geometric forms with respect to the following expression

$$a_1R_1 + ... + a_mR_m + b_1r_1 + ... + b_nr_n + c_1h_1 + ... + c_kh_k = S,$$

defined by Maja Petrović in [1] where $R_1...R_m$, $r_1...r_n$, $h_1...h_k$ are the surface points' distances from the correspondent foci, directrix lines and directrix planes; while $a_1...a_m$, $b_1...b_n$, $c_1...c_k$, are the weight coefficients respectively; and S is the desired arbitrarily chosen parameter.

Herewith, we present particularly chosen examples, being the concrete applications in architecture, which illustrate the above-mentioned rule for their generation (Fig. 1).

Key words: focal surface, directorial surface, focal-directorial surface

MSC 2010: 51N20, 53A05

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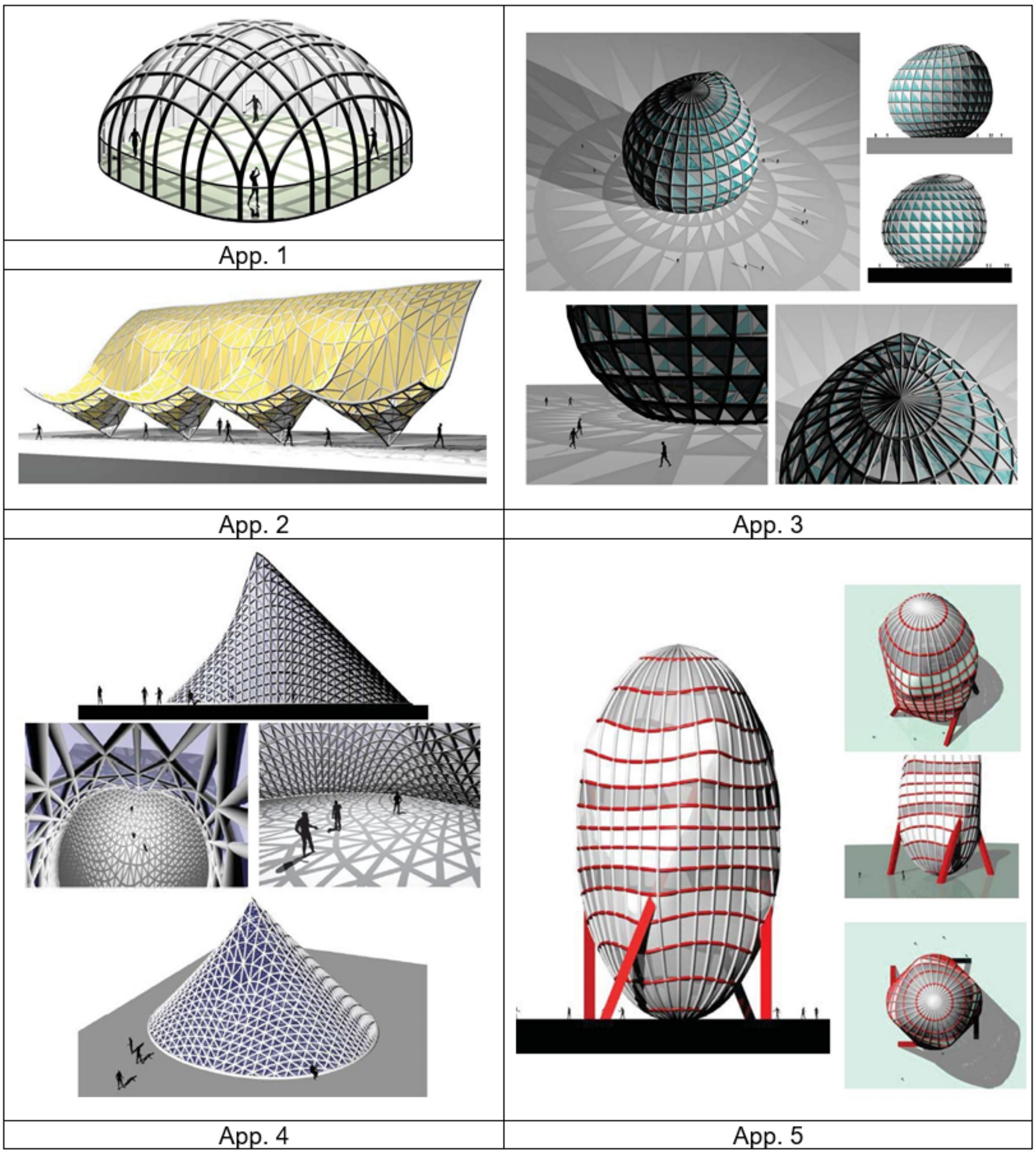

Figure 1: Focal-directorial geometric forms as a designing pattern of the architectural space

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