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ABSTRACTS

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Plenary lectures

Constructive Curves in non-Euclidean Planes

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A system of points and lines we call "geometry" if we can define a relation between them – the so-called relation of incidence – holding certain required statements. If we can define the concepts of segment and angle, respectively, then we get a "metric geometry". The equivalence of segments can be axiomatized by a new relation the so-called relation of congruency. Congruency leads to the concept of the length of a segment and the angle measure. Requiring those axioms and postulates which were used by Euclid (or later by Hilbert), we get a geometry which we call "Euclidean geometry". Contrary to the literal meaning of the phrase "non-Euclidean geometry", in the contemporary mathematics came out four geometries called non-Euclidean geometries which are on the one hand very similar to the Euclidean geometry (there are several Euclidean statements can be adopted to them), on the other hand they are characteristically different from it (there is a very important Euclidean property which does not hold in them). These geometries, derived from the absolute geometry, are the so-called hyperbolic geometry; the metric geometry of the sphere which is called spherical geometry; the geometry of the space-time or the Minkowski geometry (as it is named by physicist, but also named by pseudo-Euclidean or Lorentzian geometry); and the geometry of the finite dimensional, separable, Banach spaces which is also called Minkowski geometry by geometers.

It is more convenient to give a metric geometry using the linear algebraic approach and to show a common construction of them. Our purpose is to give a short review of the common and distinct properties of two types of constructive curves defined on the above mentioned four non-Euclidean planes, respectively. Although there are several results on such curves, there is no such summary which would compare the basic statements proved during the long period of the past two hundred years. We would like to show a common didactic approach of these planes on the base of the axiomatic method. The axiomatic connections between the geometries imply such methods which helps to adopt some theorems successfully from one non-Euclidean geometry to another one. The metric connection between any two geometries also helps us to discover well-usable "rules" to adopt metric constructions from one to the other.

Key words: conics, roulettes, non-Euclidean planes, hyperbolic plane, spherical plane, Minkowski planes

MSC 2010: 51M10, 51M15, 46C20, 53B40, 15A21

Geometric Concepts in Some Special Classes of IM–Quasigroups

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An idempotent medial quasigoup is a quasigroup which satisfies the identities of idempotency and mediality. Some geometric concepts can be introduced in each idempotent medial quasigroup. However, the definition of these concepts is given only in an implicit way. In some idempotent medial quasigroup with additional identities, these concepts could be defined in an explicit way. A number of geometric concepts in some special classes of idempotent medial quasigroups will be considered in this presentation. Specially, we are going to consider geometric concepts in a general GS–quasigroup and a quadratical quasigroup. We will study some simple geometric concepts whose existence will allow us to construct some more complicated geometric concepts. We will present the construction of an affine fullerene C_{60} in a general GS–quasigroup.

Key words: idempotent medial quasigroup, affine fullerene, GS–quasigroup, quadratical quasigroup

MSC 2010: 20N05

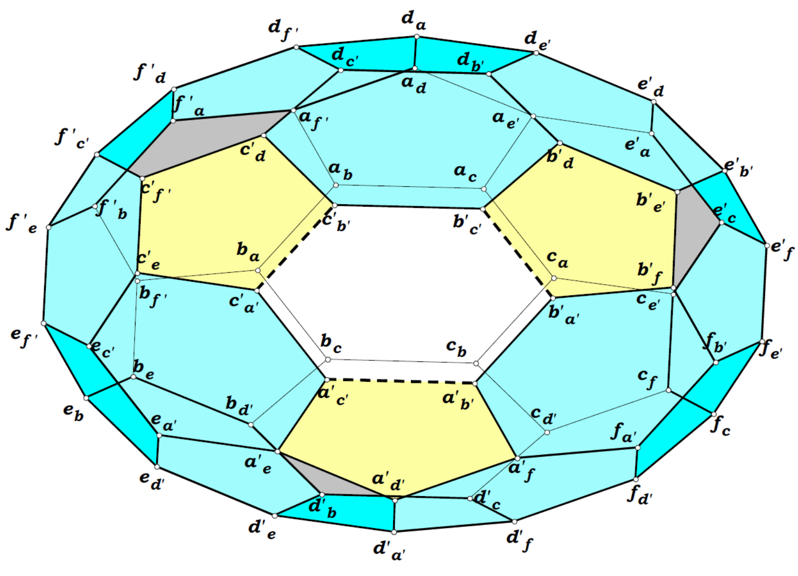

Figure 1: Affine fullerene C_{60}.

On Mathematics Education in Croatia

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Mathematics education across Europe is recognized as the educational field to be promoted and strengthened at all levels. European documents (ET 2020) address, as one of the educational targets of EU, reducing the share of low achievement in basic competences, in particular in mathematics. A project started at the Department of Mathematics, Faculty of Science, University of Zagreb, aims to address these issues and will try to enhance collaboration and flow of ideas between educational and research communities. In this talk we will present the basic ideas of this project and, as one of its research bases, results of the research conducted at the Faculty of Science which investigates understanding of mathematical concepts presented by graphs. Furthermore, we will present the short overview of curricular documents of the undergoing curriculum reform in Croatia, with a special emphasis on geometry in mathematics curriculum.

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Contributed talks

Spherical Modulation Object
A Stereotomic Design Application for a Tram Stop at the Politecnico di Milano

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This paper presents a project developed in the elective course for Master students Geometrical Complements of Graphic Representation, focusing on digital *stereotomy*. Main reference was the work of Philibert de l'Orme, born 1514 in Lyon as a son of a master mason. Due to that, he worked on construction sites from a very young age, since then being in charge of 300 masons in the town of his birth. Thanks to his experience in the construction field, he was able to focus and develop others skills, not only in geometry, mathematics and construction, but also rhythm, ornament, haptic qualities and expressivity. His joints lines are at the same time abstract and sensual - they are signs of the geometric manipulations through which it was possible to tessellate complex forms by determining the contour of the blocks. Much of his work has disappeared, but his fame remains. His masterpiece was the Château d'Anet (1552-1559), built for Diane de Poitiers, the plans of which are preserved in Jacques Androuet du Cerceau's *Plus excellens bastimens de France*, though part of the building alone remains; and his designs for the Tuileries, begun by Catherine de' Medici in 1565. We took inspiration from his work, which is crucial to our project as an example of direct link between Geometry theory and practice. His legacy could be also reflected in the architectural field nowadays, allowing us to follow his steps by complementing it with the modern software/knowledge to achieve and develop new forms (Figure 1).

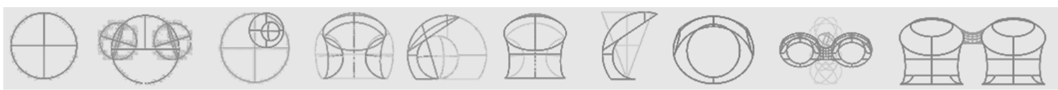

Figure 1: 2D Models - of the main steps–additions and subtractions–of the tram stop.

During classes we were requested to work on different solutions for a Tram-Stop Pavilion, located at the Politecnico di Milano, based on stereotomic principles. Having the inventions of Philibert de l'Orme in mind, we were focusing on interactions between spheres - the result should not be any random shaped form, rather it should explain itself with a well thought arrangement of geometrical forms (Figure 2).

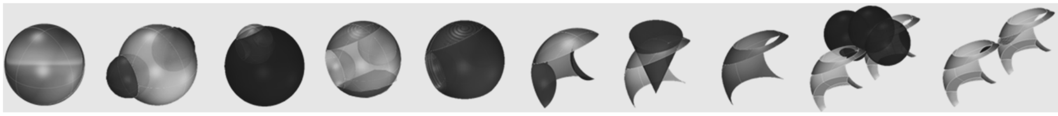

Figure 2: 3D Models - of the main steps–additions and subtractions–of the tram stop.

The first ideas lied in constructing a unique shape, though they dissolved in cause of the appearance of different tasks during our development process. Tram-Stop arose more than a shelter - beginning with issues as well-thought measurements and contextualizations showed up to be the simplest - considering the trees (with its location and altitude/amplitude), vehicle heights, the people's motion and devices to protect them of traffic/wind/rain, we were forced to deliberate. Those aspects led us to the decision of using smaller modules, each created out of an own main sphere, divided by several. Any module was connected with the other afterwards to mirror our first idea of a whole object. The solution answers its purpose as trees remain on their original positions, while people can wait for the trams - sitting/standing - protected from rain and wind.

Concerning the tessellation, the idea behind was to generate a suitable look by reusing existing objects of the module. For this we used the cones that open the top hole, and project their intersections along the whole module. By joining and erasing parts we finally found the displayed shape (Figure 3).

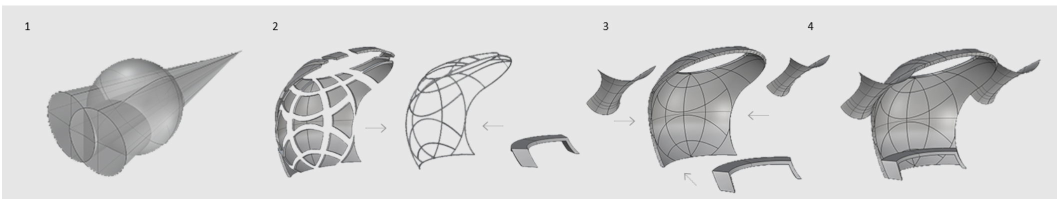

Figure 3: 1) Cones and spheres, basic origins, 2) Tessellation plus initial structure, 3) Model exploded, 4) Front view.

Key words: descriptive geometry, architectural geometry, digital stereotomy, spheres, module

MSC 2010: 00A66, 51N05, 01A05, 97U99

Figure 4: 3D Models – Final tram stop – different views: right, front, plan view, and tessellation detail.

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The Assessment of a Creative Design Course for the Development of Spatial Reasoning

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Visual-spatial skills are present in a wide range of everyday activities and are linked to success in science, technology, engineering, and mathematics (STEM) fields, [4]. In the age of Computer Aided Design, visual literacy has become increasingly important for engineers to properly interpret and represent three-dimensional objects and situations. More large-scale Hungarian studies revealed difficulties in the development of spatial skills of 18-23-year-olds, [1]. Hungarian studies correlate with similar international research results showing the high impact of spatial skills deficiencies in the failure in engineering studies [3].

The aims of our research were to develop a course which would help students overcome their deficiencies in spatial reasoning, and to investigate the background variables affecting spatial skills and methods with which these skills can be enhanced. Performance assessment tests introduced to measure the effects of the development involved four components: (1) mental analysis, synthesis; (2) mental rotation; (3) mental transformation; (4) visualisation. This presentation introduces our process-oriented teaching method and the survey instruments: questionnaires and the Hungarian Spatial Ability Test by Séra, Kárpáti and Gulyás, [2].

Analysis of results identified factors that influence the development of spatial perception and processing: the effect of learning environments (real and virtual spaces), teaching methods used in the program, secondary education, the specializations (studies for an architecture or civil engineering degree) and gender differences in performance. Our research has proved that a program based on creative problem solving tasks may develop spatial skills with at least the same efficacy as traditional methods based on drill-like exercises in representational conventions.

Key words: spatial skills, engineering education, cognitive development, 3D modelling

MSC 2010: 51N05

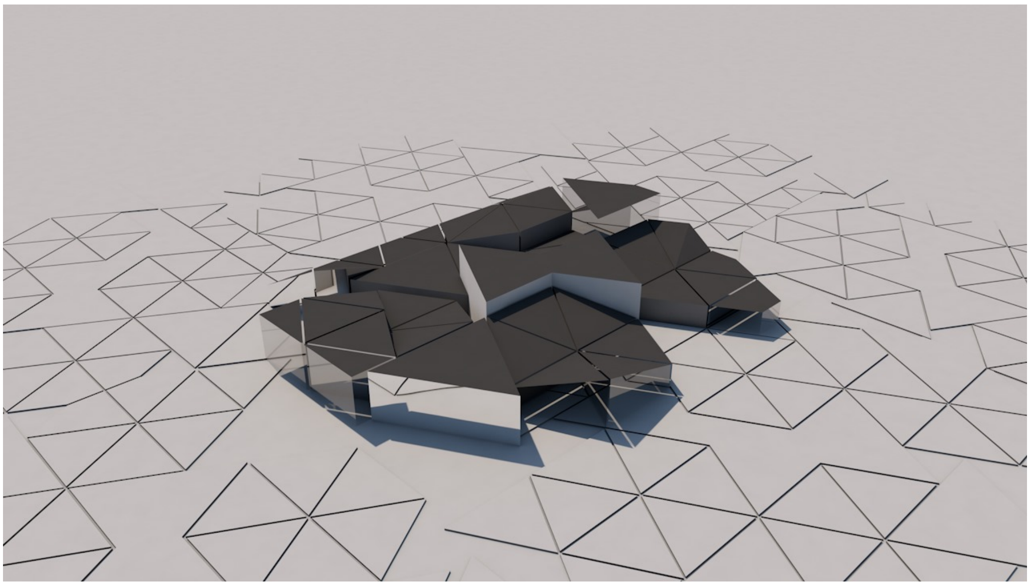

Figure 1: Spatial modeling in virtual space by Dóra Homolai, architecture student, 2016.

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Adjusting Curvatures of B-spline Surfaces by Operations on Knot Vectors

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The knot vectors of a B-spline surface determine the basis functions, hereby with the control points together the shape of the surface. Knot manipulations and their influence have been investigated in several papers (see, e.g., [2] and [3]). The computations can be made very efficiently, if the basis functions and the vector equation of the B-spline surface are represented in matrix form (see [4]). In our latest work [1] we summarized the knot manipulations and the corresponding computations in matrix form. We also developed an algorithm for a direct knot sliding, how a knot can be repositioned in one step instead of inserting a new knot value, then removing an old one from the knot vector.

In this paper we analyze the effect of sliding knots on the Gaussian curvature of a B-spline surface at a given point. In our examples we start from a B-spline surface of degree 3×3, and we modify it by giving an inner interpolation point of the surface with a prescribed Gaussian curvature value. If we move a knot value within a knot interval, then the Gaussian curvature will change between two limits. This connection can be represented numerically by a scalar function determined by properly selected pairs of (knot value, Gaussian curvature). The value of the Gaussian curvature can be prescribed within their limits (i.e., not completely freely), and it determines the corresponding knot value. If in a symmetric control net the surface is symmetric in the two parameters (i.e., the two knot vectors and the sliding knot values are equal), then the actual interpolation point chosen in the middle is a spherical point. In this case a tangential sphere can be computed with the same Gaussian curvature (shown in Figure 1).

We have described the surface in matrix form according to the non-uniform knot vectors. We have formulated the interpolation problem by linear equations for the appropriately chosen control points. We have avoided the solution of non linear equations arising by the Gaussian curvature applying a simple iteration from a scalar function. The computations and the figures are made with Wolfram Mathematica.

Key words: B-spline surface, matrix representation, interpolation, knot sliding

MSC 2010: 65D17, 65D05, 65D07, 68U05, 68U07

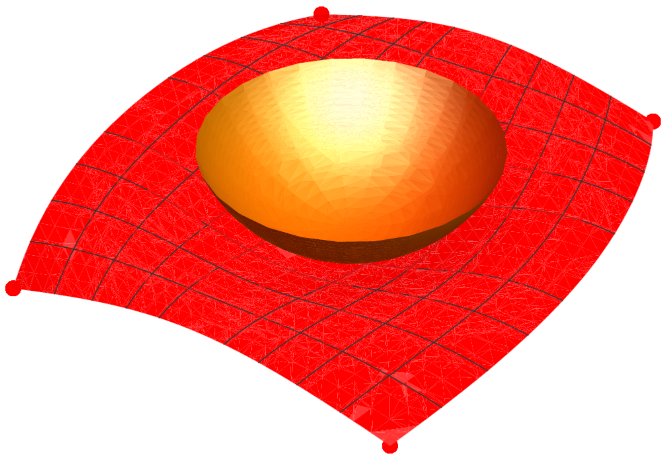

Figure 1: Tangential sphere corresponding to the Gaussian curvature at an interpolation point in the middle of the B-spline surface.

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Introduction to the Planimetry of the Quasi-Elliptic Plane

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The quasi-elliptic plane is one of nine projective-metric planes where the metric is induces by the absolute figure is $\mathcal{F}_{\mathbb{QE}} = \{j_1, j_2, F\}$ consisting of a pair of conjugate imaginary lines j_1 and j_2, intersecting at the real point F.

Some basic geometric notions, definitions and selected constructions in the quasi-elliptic plane will be presented.

Key words: quasi-elliptic plane, perpendicular points, central line, qe-conics

MSC 2010: 51A05, 51M10, 51M15

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A Glance at *Stereotomy*: a Key Step from Projective to Computational Cast

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Looking at *stereotomy*, here referred to *stone cutting* according to the Greek words *στερεός* (solid) and *τομή* (cut), it is clear its propelling role in enhancing applications and researches either in architecture or in geometry and graphics. In relation to architecture, this very special construction process has not only led Renaissance and Baroque buildings to their highest virtuosity, but has also contributed to prepare architects and engineers to the new design approach later developed during the Industrial Revolution. *Stereotomy*, indeed, promoted closer relationships between *statics* and *aesthetics* in architecture, at the point that constructive lines, that is, the contact lines among ashlars were directly exhibited as a decorative apparatus. Therefore, if on the one hand this profound integration between construction and decoration in the body of architecture can be considered as the final evolutionary stage of Renaissance and Baroque styles, on the other hand the attentive subdivision of spatial shapes into elements for constructive purposes, traditionally used also in timber structures, can be seen as an anticipation of the later prefabrication methods appearing with the use of iron and steel in building construction. In relation to geometry and graphics, such a refined approach needed for a detailed metric control of each stone and group of stones. At the end two geometrical levels had to be mastered: whole *configuration* and *tessellation*. In the Middle Age, guidelines for cutting stones were traced, directly on the construction site, normally on the ground, in order to obtain ashlars from rough stones, a method called *equarrissement* (consisting of subtractive cuts) and based on physical orthographic procedures. During Renaissance, starting from Leon Battista Alberti, the increasing distance between *drawing table* and *construction site* required a higher graphic control, encouraging significant advancements in the graphic procedures, what is clear in the new method adopted by the builders, called *par panneaux* (boards development) and based on the preliminary representation of the true-to-size graphic development of all the faces of the ashlars. Far from the illusionary purposes of perspective, projections were here finalized to the remote metrical control of the space, and the related graphic procedures were definitely opening the way to the final codification of the Orthographic Projections, in other words, focusing on *computing by images* (technical drawing) instead of on *portraying by images* (pictorial drawing). In our opinion, this background is also at the origin of the present paradigm switch from the analogue projective cast to the digital *computational cast*. Indeed, traditional sculptural operations like cutting, moving, assembling, and so forth, are nowadays graphically sublimated by the corresponding Boolean digital commands, as well as geometrical operations like configuring, morphing, and measuring spaces, can be visually performed via digital algorithms. In addition, thanks to the synergies with digital fabrication, also the gap between project and construction is finally going to be filled. Given the well-known wide spectrum of opportunities for research, a field

in which a great deal of dedicated software is under constantly increasing development, significant consequences follow in education as well. After being considered for long decades as an obsolete chapter in the academic programmes, *stereotomy* is showing now under new light as a fundamental educational topic in the field of *architectural geometry*, a starting point for building up and training students' spatial skills, together with being an efficient educational gate to attract learners and stimulate their interest about a crucial phase of the architectural history. Connected with the abovementioned issues are the expected effects on the professional world, either in relation to the re-discovery of traditional procedures for an appropriate maintenance and restoration of architectural heritage by revitalizing in a new way a disappearing professional area, or referring to the power of *stereotomy* as a secure and precious background also for those who operate in the field of modern architecture and construction with new tools, materials and techniques. Leaded by the Università Iuav di Venezia, a research unit has been set up at the Politecnico di Milano on a national research project about *stereotomy* currently being evaluated by the Italian Ministry, while an Elective Course for Master students has been activated by the author on the topic.

Key words: digital stereotomy, computational design, descriptive geometry, projective geometry, architectural geometry

MSC 2010: 00A66, 51N05, 01A05, 97U99

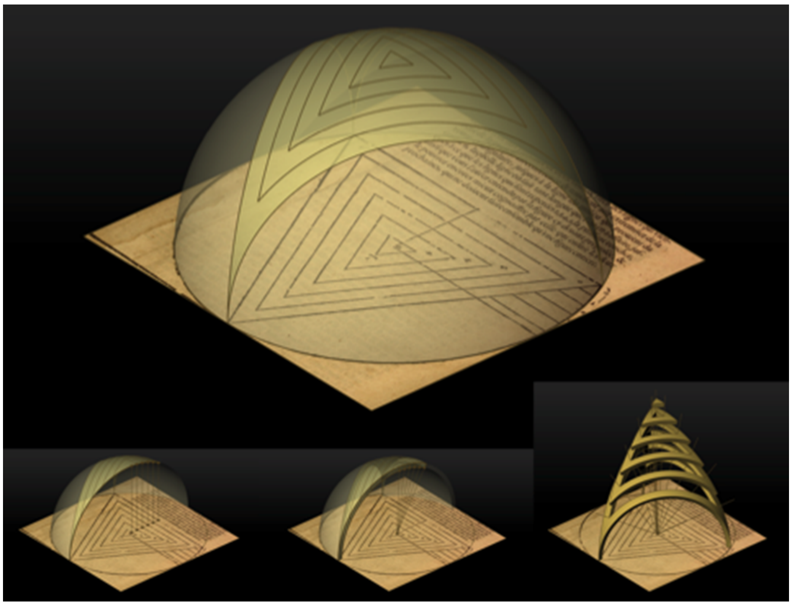

Figure 1: *Configuration* and *Tessellation*: digital remastering of a vaulted structure from the Philibert de l'Orme's treatise (1648), Livre IV, Ch. 11, p. 117 (set by the author, digital models by Matteo Cavaglià).

Acknowledgments: I would like to address my thanks to Prof. Agostino De Rosa (Università Iuav di Venezia) for involving me in applying for a national research project on *stereotomy*, and to Prof. Maria Grazia Sandri (Politecnico di Milano) for the many advices about historical sources.

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Triangles in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ Geometry

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We study the interior angle sums of translation and geodesic triangles in the universal cover of $\mathrm{SL}_2(\mathbb{R})$ geometry. We prove that the angle sum $\sum_{i=1}^{3}(\alpha_i) \geq \pi$ for translation triangles and for geodesic triangles the angle sum can be larger, equal and smaller than π.

Key words: Thurston geometries, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ geometry, triangles, spherical geometry

MSC 2010: 52C17, 52C22, 52B15, 53A35, 51M20

Geometrical Measurements in Learning Analytics

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Learning analytics is focused on the educational challenge of optimizing opportunities for meaningful learning and teaching.

Some metrics and measures used in learning analytics can be considered and justified from geometrical point of view. Some of the metrics can be modified and tested on real data produced in from learning and teaching environment.

Assessment is of crucial importance as it deeply influences learning or other processes that are assessed. At the same time data about assessment are still rarely considered and utilized by learning analytics. Assessment analytics are an area in the field of learning analytics especially fruitful for geometrical modelling. Several important notions such as reliability of assessment can be interpreted in geometrical ways by use of Euclidean as well as non-Euclidean geometries.

Our proposal addresses these issues with a geometrical model based on the taxicab geometry. A geometrical model is presented and discussed for two important peer-assessment scenarios. Finally, we present empirical results based on data from Learning Management System.

Key words: taxicab geometry, learning analytic, geometrical modelling

MSC 2010: 53A35, 91E45, 97B10

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Contact Constant Angle Surfaces in Sol_3

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In classical differential geometry of surfaces in Euclidean 3-space $\mathbb{E}^3$, a surface M is said to be a *constant angle surface* if its unit normal vector field makes constant angle with a fixed direction.

Constant angle surfaces have been investigated in the following ambient spaces: the product space $\mathbb{S}^2 \times \mathbb{R}$ [1], $\mathbb{H}^2 \times \mathbb{R}$ [2], warped products of the form $\mathbb{R}_f \times \mathbb{R}^2$ [3], the Heisenberg group Nil_3 [4], the model space Sol_3 of solve geometry [5, 8] and the solvable Lie group model of the hyperbolic 3-space $\mathbb{H}^3$ [8], respectively. For modern treatments of constant angle surfaces, we refer to [6, 7, 8].

The space Sol_3 admits a contact structure compatible to the metric. Thus the Reeb vector field of the contact structure naturally defines a fixed direction.

In this talk we consider surfaces in Sol_3 which make constant angle with Reeb vector field.

Key words: *Sol* geometry, constant angle surface, Reeb vector field

MSC 2010: 53A10, 53C15, 53C30

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Quadrics of Revolution on Given Points

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There are very early contributions of E. Laguerre [4] to the task of finding all quadrics of revolution on 5 given points P_1, P_2, P_3, P_4, P_5: Laguerre interlinked this problem to the circumspheres determined by the 5 quadruples P_i, P_k, P_j, P_l that can be established from the five given points. More recently, the problem of determining all right cylinders on 4 given points P_i has been solved by H. Schaal [10], [11]: There is a 1-parametric set of cylinders of revolution and in case of non-coplanar points P_i their axes establish an algebraic surface of degree 3. The task of finding all right cylinders on 5 points is an algebraic problem of order 6: In the generic case there exist 6 solution cylinders at most [14]. The cones of revolution on 4 given points establish a 2-parametric set, the locus of vertices is an algebraic surface of order 14, see [12], [13]. O. Röschel [9] determined all quadrics of revolution containing a given conic section. There are also early contributions to this problem by A. Narasinga Rao and M. S. Srinivasachari [7], [8].

In my presentation I discuss the approach [3] which allows to identify different kinds of quadrics of revolution on a given number n of points. By means of this method one can easily check that the axes of the 1-parametric set of quadrics of revolution on 6 prescribed points intersect the plane of infinity along a conic section. Moreover I will show that there are at most 4 quadrics of revolution on 7 given points and at most 12 right cones on 6 given points in the generic case. The forementioned task of determining the right cylinders on 5 points can also be treated by the introduced method. In each case I can give examples where the maximal solution numbers 4, 12 and 6 are obtained by *real* quadrics of revolution, right cones and right cylinders, respectively.

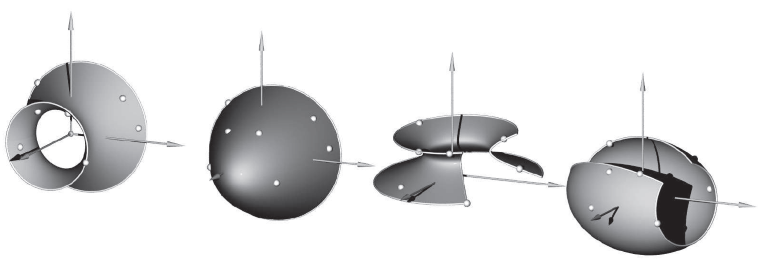

Figure 1: Four quadrics of revolution on seven given points.

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From Lascaux over Fibonacci to the present: Geometry - Mathematics - Nature - Art

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Mathematics and geometry have a lot to do with nature, even if one cannot always see this as directly as, for example, in the Fibonacci numbers. Nature is pragmatic and accepts solutions that arise from selection or random constellation, if the solution is better than a previously existing one. This applies to the development of living beings as well as to the formation of shapes or patterns. In return, mathematics and geometry have undergone an evolution in which the understanding and description of natural processes has played an important role. On the one hand, mathematics is a self-contained cosmos.

Can mathematics and geometry also be related to art? The fact is that, after a period of rather less agreement on both sides, the focus is once again, just as in the Renaissance, on what unites these disciplines. Not every work of art has to do with mathematics or geometry and vice versa. Yet, there are a number of beautiful examples throughout human history in which common ideas can clearly be seen. Some of these will be discussed in the lecture.


By moving a sphere along a generating line, it sweeps up a so called pipe surface (or a tubular surface), which comes close to the shape of a millipede -- no matter if it's curled up or not.
Despite the name, no known millipede has anywhere close to 1.000 legs. In fact, it has two pairs of legs on each segment, so it is probably easier to count the segments and multiply this number by four.
Computer graphics simulation of a helical pipe surface
Astronomy was always a typical field of application for geometric theory. Monumental structures like Stonehenge and the Egyptian or Central American pyramids served as giant observatories for the evidently strange motions in the sky. The alignments of those structures were not only influenced by sun positions, but might also have been arranged with other aspects in mind, such as the puzzlingly extreme moon positions (one might think of lunar standstills) and the special positions of the Pleiades.
Some special constellations allowed ancient societies on three different continents to precisely determine long periods of time with nothing more than Stone Age methods -- for instance, the length of a year or the time in between lunar standstills. Coincidentally, a Venus-year is almost exactly 5/8 of the sidereal year. Therefore, when the Mesoamerican astronomers observed the "August-13-phenomenon" (Pleiades at the zenith during sunrise), they also realized that in every eighth year Venus was at the same relative position in the sky.
⊥ to Pleiades in zenith
⊥ and ∥ to equinoxes (Kukulkan)
∥ to summer solstice
⊥ to sun in zenith
Some temples and platforms in Chichen Itza precisely indicate important sunset directions (Google Earth image)
Temple of Kukulcan in Chichen Itza
How to transport heavy objects up a pyramid by means of manpower
The shadow of the stairs of the Temple of Kukulcan was watched carefully. During equinoxes, it resembled a „Feathered Serpent" (the symbol of Kukulcan).
The Möbius strip is a surface with only one side and only one boundary component and therefore cannot be oriented. It is rather easy to produce: Take a rectangular paper strip, twist it and join the ends together to form a loop (see the copmuter graphic). Depending on whether you twist clockwise or counterclockwise, you get a right-handed or left-handed strip.
Klaudia Kozma is fascinated by Moebius strips and uses them to create all kinds of artworks. She has developed a method of generating an arbitrary number of strips out of a single one by means of refinement. The only limitation is the material. The strips are partly circular and may even have „corners". In the absence of corners, some of the strips fold themselves quite naturally above the base plane without the need of twisting them (left image).

Figure 1.

Graph Colouring and its Application within Cartography

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The problem of colouring geographical political maps has historically been associated with the theory of graph colouring. In the middle of the 19th century the following question was posed: how many colours are needed to colour a map in a way that countries sharing a border are coloured differently. The solution has been reached by linking maps and graphs. It took more than a century to prove that 4 colours are sufficient to create a map in which neighbouring countries have different colours.

In graph theory, graph colouring is a special case of graph labelling. It is about assigning a colour to graph elements: vertices, edges, regions, with certain restrictions.

With this presentation we would like to assess the elements of the theory of graph colouring with an emphasis on its application on practical problems in the field of surveying, namely cartography.

A mathematical basis for administrative map colouring will be given along with the chronology of proving *The 4 Colour Theorem*. In addition, World maps and map of Croatia will be shown, to determine the minimum number of colours needed to colour a map properly in practice.

Application

Cartography is a discipline that deals with collecting, processing, storing and usage of spatial information, being especially concerned in their visualisation i.e. cartographic representation. A *map* is a connected planar graph where all vertices have a degree of at least 3. It is a unit consisting of interconnected regions/countries. The border of each region represents a closed curve that can be divided into as many parts/edges as the region has neighbours. The two countries that share a common edge are considered *adjacent*.

As well as being a tool that is used for better orientation in space and for travel from point A to point B, maps also possess an artistic component. Each colour causes a certain stimulus therefore it is necessary to pay special attention to the choice of colours and their diversity when producing a map. Selecting the optimal number of colours is also valuable because it enables cheaper reproduction of maps.

Key words: graph, graph colouring, map, map colouring, 4 colour theorem

MSC 2010: 05C15, 05C90, 86A30, 68R10

Folk Ornament as an Important Factor in Identifying a Plane Geometric Shape

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The paper presents the possibility of developing geometric competencies through folk culture. Our attention is focused on a particular folk ornament that can be used for presentation and identification of a plane geometric shape by pupils during the teaching of mathematics. We will focus on the possibilities for the analysis of geometric shapes in a variety of ornaments used in folk culture, in different textile and functional objects. We will describe the specifics of primary education in Slovakia with regard to the interdisciplinary context.

The geometry teaching has an important place in the Slovak state curriculum ISCED1 for primary education. Some goals for the first class in primary school are the following:

- to identify, to name and to draw an open and closed, curved line,
- to identify, to name, to outline a straight line,
- to identify, to name, to draw a plane geometric figure.

It means to teach pupils planar geometric shapes: curved line, straight line, open and closed line, circle, square, triangle, and rectangle. It is possible in this art of teaching to use components of folk culture as an interdisciplinary element. Many geometrical shapes can be seen in the picture below.

Figure 1: Some geometrical shapes in textile

Key words: plane geometric shape, geometry teaching according Slovak state curriculum ISCED1, ornament in textile

MSC 2010: 97D40

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Visualization and Clustering of Population Pyramids

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Population pyramids are a type of histograms. They present the age distribution of a population, also taking gender into consideration. Two gender histograms are plotted horizontally back-to-back. The number of males/females or the corresponding proportion is presented on the x-axis and age-groups are presented on the y-axis [1].

Our presentation has two goals. The first goal is to demonstrate a possibility for clustering of data suitable for presentation in the form of a histogram, i.e. to find units (or in our case populations) that are similar by structure to each other. Similarity or distance of the items has to be defined by a distance measure. There are wide range of measures that can be used [2]. Among others, we will use the Euclidean distance. The second goal is to demonstrate a possibility for graphical visualization of the population pyramids and the geo-spatial presentation of the clustered data using R programming language [3].

As an example, we use census data from the Croatian Bureau of Statistics for 20 Croatian counties and the City of Zagreb for the years 2001 and 2011 [4]. Visualization and clustering of counties directs attention on change in populations' age structure and indicates that pattern of population aging is different among counties in Croatia.

Key words: clustering, data visualization, demographics structure, population aging

MSC 2010: 91C20, 62-07, 62-09

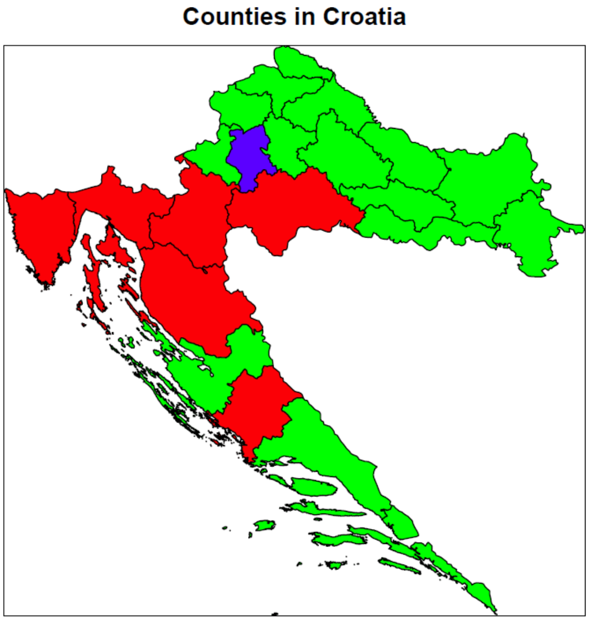

Figure 1: Counties in Croatia clustered by population age structure.

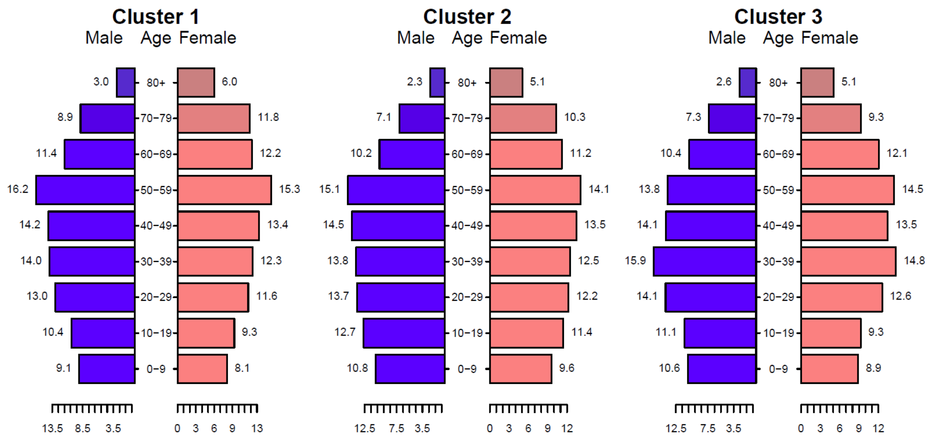

Figure 2: Population proportions pyramids - Croatia 2011 - clustered counties.

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Generalized Computable Intermediate Value Theorem

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Let $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. The intermediate value theorem states that if u is any number in the interval $\langle f(a), f(b)\rangle$, then there exists a number $c \in \langle a, b\rangle$ such that $f(c) = u$. As a consequence of this theorem we get a result, which states that if $f\colon [0,1] \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(0) < 0$ and $f(1) > 0$, then it has a root in the segment $[0,1]$. This consequence of the intermediate value theorem is known as Bolzano's theorem.

Let us now take a computable function $f\colon [0,1] \to \mathbb{R}$ such that $f(0) < 0$ and $f(1) > 0$. Then it is well known that there exists a computable number $x \in [0,1]$ such that $f(x) = 0$ [7]. This is a computable version of the intermediate value theorem. If we denote by $S = \mathbb{R} \times \{0\}$ and by K the graph of a computable function $f\colon [0,1] \to \mathbb{R}$, i.e. $K := \Gamma(f) = \{(x, f(x)) : x \in [0,1]\}$, then f has a computable zero-point if and only if K intersects S in a computable point. Since S and K are computable sets in $\mathbb{R}^2$, the question which arises is the following: under what assumptions the intersection of two computable sets contains a computable point?

There exists a nonnegative computable function $f\colon [0,1] \to \mathbb{R}$ which has zero-points, but none of them is computable [8]. Note that for such a function f we have $K \cap S \neq \emptyset$, but K does not intersect S in a computable point. Also note that the complement of S in $\mathbb{R}^2$ has two connected components - the upper and the lower open half-plane and the graph of f does not intersect the lower one.

Suppose that K is a connected subset of $\mathbb{R}^2$ which intersects both components of $\mathbb{R}^2 \setminus S$. Then K certainly intersects S. In view of previous fact, the question is does the following implication hold:

$$K \text{ computable} \Rightarrow \; K \text{ intersects } S \text{ in a computable point.} \tag{1}$$

The following example shows that previous implication does not hold in general. Let $f\colon [0,1] \to \mathbb{R}$ be a nonnegative computable function wich has zero-points, but non of them is computable. Let $K = \Gamma(f) \cup \Gamma(-f)$. Then K is a connected and computable subset of the plane, it intersects both components of $\mathbb{R}^2 \setminus S$, but K does not intersect S in a computable point.

However, there are possibly some additional assumptions under which (1) holds. We examine certain conditions under which (1) holds even in a more general context and we show result which generalizes the case when $K = \Gamma(f)$, where $f\colon [0,1] \to \mathbb{R}$ is a computable function such that $f(0) < 0$ and $f(1) > 0$.

We generalize the ambient space. Suppose (X, d, α) is a computable metric space. Let U and V be disjoint and computably enumerable open sets in this space and let $S = X \setminus (U \cup V)$. Suppose $K \subseteq X$ is a chainable continuum which intersects U and V. We prove that (1) holds under assumption that $K \cap S$ is totally disconnected. This result will imply the following: if A is a computable arc which intersects U and V, then A intersects S in a computable point. Because the graphs of computable functions $[0, 1] \to \mathbb{R}$ are not just any computable continua, they are computable arcs, we have generalized computable version of intermediate value theorem.

We will also examine the case when $K \cap S$ is not totally disconnected. It turns out that this case is much more difficult, and for now we have proved that (1) holds under assumption that $K \cap S$ has isolated and decomposable connected component.

Key words: computable metric space, chainable continuum, computable compact set, computably enumerable open set, computable point

MSC 2010: 03D78

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On Nontriangulable 4-dimensional Polyhedra

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Triangulations of 3-dimensional polyhedron are partitions of the polyhedron with tetrahedra in a face-to-face fashion without introducing new vertices. We generalize this problem to 4 dimensions and we construct nontriangulable 4-dimensional polyhedra.

Key words: polyhedron, simplex, nontriangulability

MSC 2010: 52A15, 52C22

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Some Simple Geometric Simulations

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In many cases practical geometrical problems can be solved easier and faster by software simulations than by exact calculations. (For example there are several engineering design standards and recommendations for planning road geometry, but they do not contain special and extreme cases.) We can use specific software programs for geometric simulations, but they are often too complicated and specialized.

Dynamic Geometry Software (DGS) and Computer Aided Design (CAD) applications are universal and rife. They are convenient for fast sketching and demonstrating practical geometric problems. They allow to define geometric constraints, and to animate our constructions. In this presentation we show some real practical problems and solutions by simulation, which are suitable for introducing our students to this method.

Key words: geometric simulation, geometric constraint

MSC 2010: 068N30, 97G99

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Geometric Representation of η-quotients

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A class of functions named η-quotients consists of functions obtained by rescaling and multiplying the Dedekind η-function. This function was introduced in 1877 by R. Dedekind. Under certain conditions, η-quotients are modular forms for the congruence subgroup $\Gamma_0(N)$, i.e., holomorphic functions of the complex upper half-plane with certain transformation properties with regard to the group action of $\Gamma_0(N)$ on the upper half-plane. We will show the geometric interpretation of these conditions.

Key words: η-quotients, modular forms

MSC 2010: 11F20, 11F11

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Spatial Thinking and STEM

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In recent years, there has been much talk about the importance of STEM fields (Science, Technology, Engineering and Mathematics) in pupils' preparations for technological and economical challenges of the 21st century.

In this presentation the focus will be given, according to the results of several studies, onto one of the crucial factors that can influence pupils' ability to acquire and practice knowledge from STEM-related fields: spatial thinking. With analyzing the link STEM learning and spatial learning, special attention will be given to some obstacles to learning spatial thinking, within the geometrical thinking, which includes understanding and developing several cognitive processes included in spatial thinking.

Furthermore, the examples of several project tasks will be highlighted, based on psychological, theoretical and practical considerations, the application of which is expected in the coming school year within the optional subject of one secondary school in Croatia as a part of the project: *Modern technologies and training methods for the acquisition of skills and competencies in high schools: STEM for everyone.*

Key words: spatial thinking, geometry of space

MSC 2010: 97C30, 97C70, 97G40, 97G80

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Correlation Between Mental Rotation as one of the Spatial Abilities and Rotation as Individual Procedure in Descriptive Geometry

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Spatial intelligence, which includes spatial perception, is an integral part of human ability. The study of spatial perception is a relatively young discipline. The expression "spatial perception" covers a wide area comprising a range of skills.

The Faculty of Architecture has tested freshmen for spatial perception every year since 1999. The level of students' conception has been monitored through the Descriptive Geometry course, using the mental rotation test (MRT). The test excellently covers the field of mental rotations rather than other fields. We set up the hypothesis that the individual procedure dealing with rotation is complex in student performance used in descriptive geometry. We wanted to link these finding with the every year verifying students' spatial conception using the MRT test.

In order to analyse the state of spatial conception, we analysed individual exercises of midterm exams and projects from the perspective of individual procedures required to obtain the correct result. We classified these procedures into groups, sections and fields. These procedures include spatial relations of points, lines, and planes, and some special constructions.

Student performance was the best in case of the position of lines and points. It was indicated that the procedures dealing with spatial perpendicularity, piercing, sectioning and, in particular, rotation are most complex.

With the MRT test we are verifying the area of spatial conception that is the most complex in student performance. We will continue with every year verifying students' spatial conception using the MRT test and also include more content relating to the rotation as individual procedure in the educational process at the Faculty of Architecture.

Key words: spatial ability, descriptive geometry, education

MSC 2010: 51N05

Table 1: Performance in the use of individual procedures.

			all midterm exams and projects		all, with the exception of the projects done at UL FA		midterm exams		projects	
field	*subfield*	*category*	*N*	*performance (%)*	*N*	*performance (%)*	*N*	*performance (%)*	*N*	*performance (%)*
affinity	*affinity*	*affinity*	3	56,6	3	56,6			2	47,3
coordinate system	*drawing points using coordinates*	*drawing points using coordinates*	16	59	13	55,3			16	59
	octants	*octants*								
position of a line	*position of a line on the plane*	*position of a line on the plane (except for Π_1–Π_3, symmetrical and coincident)*	11	59,7	10	58,5	4	67,6	7	55,2
		position of a line on the Π_1 and/or Π_2 plane	10	63,2	8	59,8	3	59,7	7	64,7
		position of a line on the Π_3 plane	6	66,9	4	61,9			6	66,9
		position of a line on the symmetrical and/or coincident plane								
position of a point	*position of a point on the line*	*position of a point on the line*	25	57,9	22	55,5	10	56,9	15	58,5
	position of a point on the plane	*position of a point on the plane (except for Π_1–Π_3, symmetrical and coincident)*	9	62,2	7	57,9	3	52,7	6	66,9
		position of a point on the Π_1 and/or Π_2 plane	13	65,9	11	63,9	5	69,1	8	63,9
		position of a point on the Π_3 plane	7	65,7	5	61,1			7	65,7
		position of a point on the symmetrical and/or coincident plane								
	similarity of strings	*similarity of strings on the line*								
perpendi - cularity	*perpendicularity on the plane*	*perpendicularity on the plane*	8	63,8	6	59,4			7	59,9
	spatial perpendicularity	*spatial perpendicularity*	7	52,2	6	47,2	2	34,9	5	59,2
piercing and sectioning	*piercing and sectioning (except for Π_1–Π_3, symmetrical and/or coincident)*	*piercing and sectioning (except for the Π_1–Π_3 planes, symmetrical and/or coincident)*	9	49,1	8	44,8	3	36,8	6	55,2
		piercing and sectioning through the symmetrical and/or coincident plane								
		piercing and sectioning through the Π_1 and/or Π_2 plane	17	54,2	14	49,7	3	38	14	57,6
		piercing and sectioning through the Π_3 plane	7	65,7	5	61,1			7	65,7
projection	*projection (except for Π_1–Π_2)*	*projection (except for Π_1 and/or Π_2)*	7	52,1	6	48,9	3	52,7	4	51,6
		projection on Π_1 and/or Π_2	21	57,9	19	55,9	8	54,6	13	59,9
		projection on Π_3	7	67,5	5	63,7			7	67,5
	side projections	*side projections*	6	51,6	6	51,6	2	56,3	4	49,2
rotation	*rotation*	*rotation*	5	43,9	4	37	2	26,8	3	55,3
parallelism	*parallelism of lines*	*parallelism of lines*	11	53,9	9	48,8	3	47,3	8	56,4
	parallelism of planes	*parallelism of planes*	3	41,9	3	41,9			2	44,4
		TOTAL		57,7		54,3		50,3		58,6

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Top Dense Ball Packings and Coverings by Hyperbolic Complete Orthoscheme Groups

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In mathematics sphere packing problems concern the arrangements of non-overlapping equal spheres which fill space. Usually space is the classical three-dimensional Euclidean space $\mathbf{E}^3$. However, ball (sphere) packing problems can be generalized to the other 3-dimensional Thurston geometries.

In an n-dimensional space of constant curvature $\mathbf{E}^n$, $\mathbf{H}^n$, $\mathbf{S}^n$ $(n \geq 2)$ let $d_n(r)$ be the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by the centres of the spheres. L. Fejes Tóth and H. S. M. Coxeter conjectured that in an n-dimensional space of constant curvature the density of packing spheres of radius r can not exceed $d_n(r)$. This conjecture has been proved by C. Roger in the Euclidean space. The 2-dimensional case has been solved by L. Fejes Tóth. In a 3-dimensional space of constant curvature the problem has been investigated by K. Böröczky and A. Florian in [2] and it has been studied by K. Böröczky in [1] for n-dimensional space of constant curvature $(n \geq 4)$. The upper bound $d_n(\infty)$ $(n = 2, 3)$ is attained for a regular horoball packing, that is, a packing by horoballs which are inscribed in the cells of a regular honeycomb of $\overline{\mathbb{H}}^n$. For dimension $n = 2$, there is only one such packing. It belongs to the regular tessellation $\{\infty, 3\}$. Its dual $\{3, \infty\}$ is the regular tessellation by ideal triangles all of whose vertices are surrounded by infinitely many triangles. This packing has in-circle density $d_2(\infty) = \frac{3}{\pi} \approx 0.95493$.

In $\overline{\mathbb{H}}^3$ there is exactly one horoball packing with horoballs in same type whose Dirichlet–Voronoi cells give rise to a regular honeycomb described by the Schläfli symbol $\{6, 3, 3\}$. Its dual $\{3, 3, 6\}$ consists of ideal regular simplices T^∞_{reg} with dihedral angles $\frac{\pi}{3}$ building up a 6-cycle around each edge of the tessellation. The density of this packing is $\delta_3^\infty \approx 0.85328$.

In hyperbolic space $\mathbf{H}^3$ the densest horoball packing can be realized by different regular arrangements [3]. If we allow horoballs of different types at the various vertices of a totally asymptotic simplex and generalize the notion of the simplicial density function in $\mathbf{H}^n$ $(n \geq 2)$, then the upper bound for the density of Böröczky–Florian type does not remain valid for the fully asymptotic simplices [7], [8].

In [9], [5] and [4] we have studied some new aspects of the horoball and hyperball packings in $\mathbf{H}^n$ and we have realized that the ball, horoball and hyperball packing problems are not settled yet in n-dimensional $(n \geq 3)$ hyperbolic space.

But, there are no "essential" results regarding the "classical ball packings and coverings" with congruent balls. What are the densest ball arrangements in $\mathbf{H}^n$ and what are their densities?

The goal of this talk is to study the above problem of finding the densest ball arrangements in 3-dimensional hyperbolic space with "classical balls". In this talk we consider congruent periodic ball packings (for simplicity) related to the generalized Coxeter orthoschemes. We formulate a conjecture for the densest ball packing arrangement and compute its density.

We will use the well-known Beltrami-Cayley-Klein modell of $\mathbf{H}^3$ with projective metric calculus [6].

Key words: hyperbolic geometry, ball packings, packing density, Coxeter orthoschemes

MSC 2010: 52C17, 52C22, 52B15

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Visualization and Prediction of Students' Course Progress

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Data are potential source of valuable information. Analytical techniques used on topic-related data give user deeper insights in the field and can directly support assessment, planning and decision making. Based on goals, analytics are divided into three groups: descriptive, predictive and prescriptive. Goal of descriptive analytics is to improve the understanding of the domain. Predictive analytics is oriented on the successful prediction of the future events. Prescriptive analytics provides its users advice on actions and gives guide to a desired solution. Such technics used in the area of learning and teaching are usually called learning analytics.

"The first stage of data analysis is to make sense of the data before proceeding with more goal-directed modeling and analyses" [1]. Our presentation has two goals. The first goal of our presentation is to visualize students' course progress. For successful accomplishment of the goal we need to present multi-dimensional data in a simple way and detect groups of students with similar progress pattern. Among others, logistic regression and linear discriminant analysis will be presented as methods for supervised learning, also useful for prediction of future students' progress through course [2]. Presentation of the methods is the second goal of our presentation.

Analysis and visualizations are presented using data about the students' progress during the course *Financial mathematics* for two sequential academic years, 2014/2015 and 2015/2016. Visualization of the progress data is done using R programming language and appropriate packages, [3].

Key words: data visualization, logistic regression, predictive analytics, learning analytics

MSC 2010: 62-07, 62-09, 62J02, 91C20

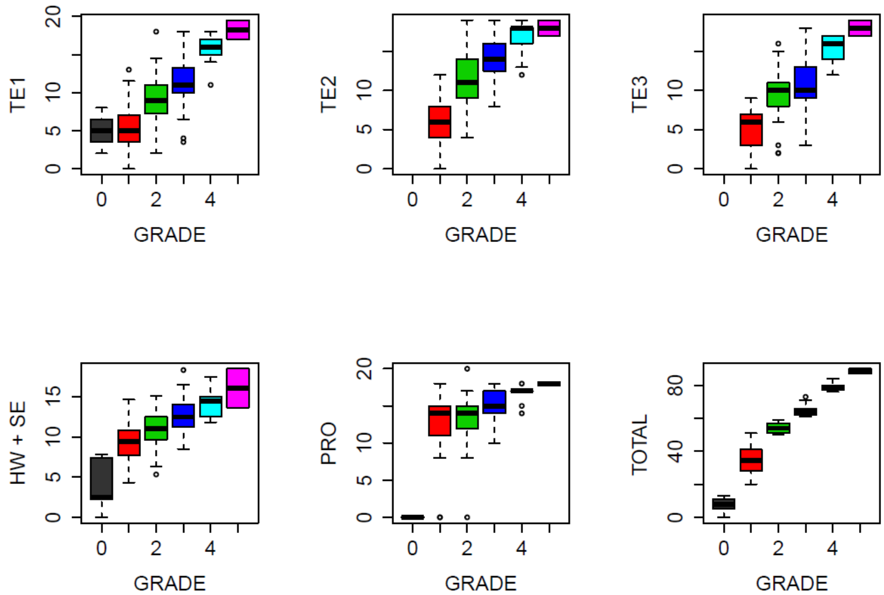

Figure 1: Boxplots of points achieved at activities conditional on final grade achieved at course *Financial mathematics* - academic year 2015/2016.

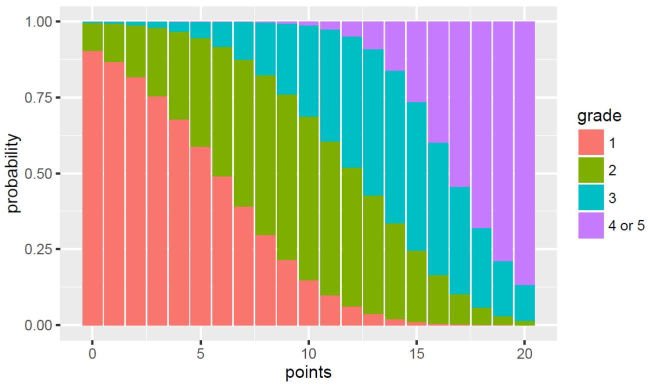

Figure 2: Estimated probability of final grade in the course based on result of the first Term-exam (course *Financial mathematics*).

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Interchange Tram-Stop: a Sterotomic Design Experience

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Interchange is a tram-stop design inspired by the vaults of Philibert de l'Orme using cut spheres, which are interchanging and creating a new spatial complex. Our first spatial reference for design was the structure of the side aisle of a typical arcade, where the spherical vaults are arranged along one axis, inspired by the Brunelleschi's Ospedale degli Innocenti (Figure 1). Transforming this old structure we interchanged the vaults into a new dynamic space, playing with the center of each sphere and moving towards each other to interlock them (Figure 3). In our model, we combined the axis of the structure of a historical reference with the existing axis of the trees. The idea was to have trees as a trace of regularity of classical space. We applied a third sphere to intersect our structure from above. This creates a void, by intersecting the two axes into a new design (Figure 4). Our design considerations were safety and spatial restriction in the urban situation in front of the building of the Architectural faculty of Politecnico (Figure 2).

We applied two different materials for each side of the tram station. The inner-side towards the tram is made out of transparent material and the outer side of the stop is made out of a bold material. The combination of different materials gives the design an extra aesthetic aspect. The stereotomic approach we were using during the course *Geometrical complements of graphic representation* led by professor Luigi Cocchiarella enabled our group to realize the power of architectural geometry nowadays. The academic design project we were developing became the strong basis of the knowledge we acquire about the process of advanced digital computational design.

Key words: stereotomy, architectural geometry, surface, sphere, vault

MSC 2010: 00A66, 51N05, 01A05, 97U99

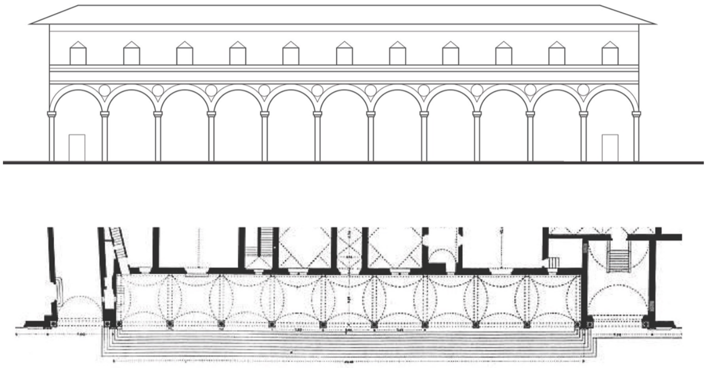

Figure 1: Reference space - arcade with spheric vaults of Ospedale degli Innocenti by Brunelleschi. Facade and fragment of the ground floor plan.

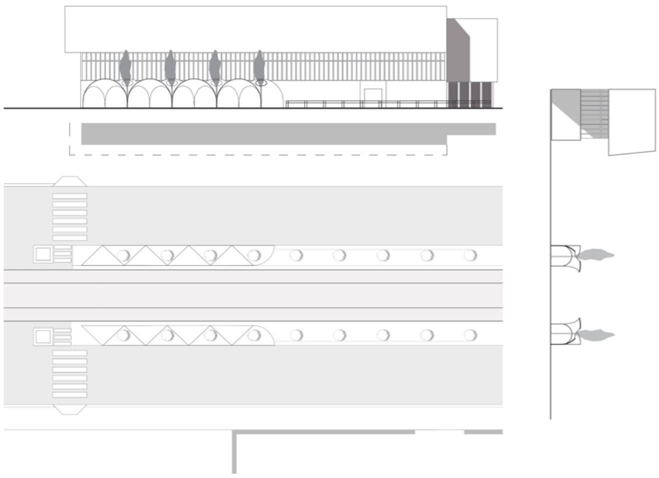

Figure 2: View of the tram-stop with Architectural Faculty building behind. Plan view with the urban situation and tram-stops of the both directions. Side view from the pedestrian crossing on the tram-stops, the Architectural Faculty building shows on the left side.

Figure 3: Dynamic space of the tram stop. View along the railways.

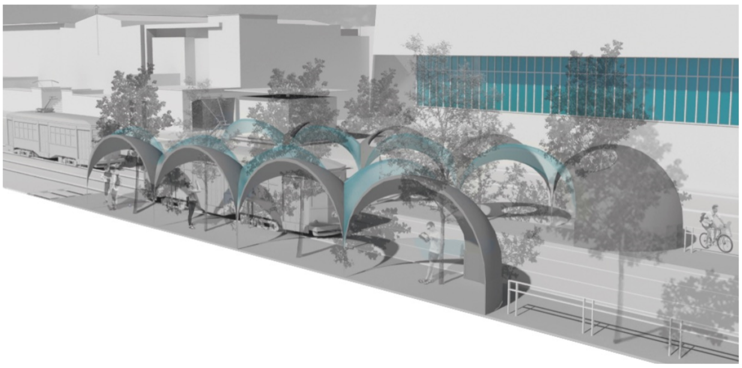

Figure 4: View on the intersecting spheres of the tram-stop structure.

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Characterization of the Motions of Fibrils

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Introducing a 3D mechanical model for the motion of crosslinked cylindrical networks under axial motion constraint, we provide a graph theoretical model and give a graph theoretical characterization (Figure 1) of the flexibility and the rigidity of this network. The edge or node connectedness of the bracing graph characterize the rigidity of these structures, notwithstanding the consideration of the circuit of the bracing graph is also important. In this presentation, we focus on the kinematical properties of hierarchical levels of fibrils and evaluate the number of the bracing elements for the rigidity. We present some conclusions, including perspectives and future developments in the frameworks of biostructures such as microtubules, collagens, cellulose, which inspired this work.

Key words: packing of the cylinders, bar-joint framework, graph connectivity

MSC 2010: 03C50, 52C25

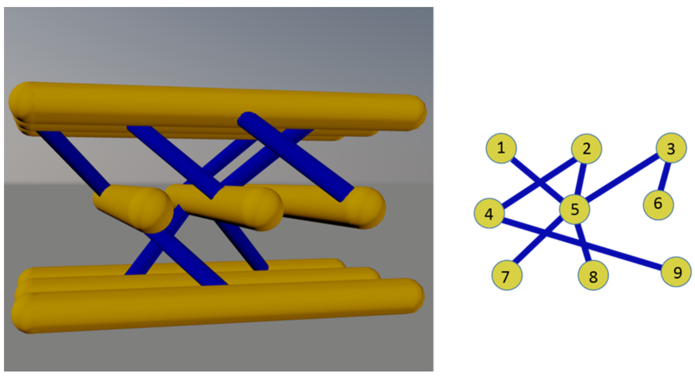

Figure 1.

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Hypercube Models Constructed on Spherical Surfaces

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The 3-dimensional framework (3-model) of any k-dimensional cube (k-cube) can be produced from any initial k edges. The Minkowski sum of these can be called zonotope. The regulated model is rotational and central symmetric, all edges have the same angle to a ground plane and are parallel with one of the initial edges. Some inner vertices and edges can be coinciding.

Our goal is to construct a similar model on sets of spherical surfaces. The outer vertices join meridians that have a rotational symmetric layout on a sphere. Two of these vertices are on the poles, and $(k-1)$ times k vertices join $(k-1)$ lines of longitude arranged in pairs symmetrically to the equator. The outer edges with the same length are segments of great circles and connect the vertices along two meridians. These arcs have the same central angle and the common centre point in the midpoint of the sphere. The angle can be defined by analytical description of lemniscates. The inner vertices join concentric spheres. The connecting inner edges are segments of great circles of the above spheres or of other spherical surfaces intersecting and touching the concentric ones. The construction of the inner structure of the models has no well defined general algorithm. It requires special decisions depending on the number of the dimensions of a hypercube.

Key words: analytic geometry, constructive geometry, hypercube, zonotope, lemniscate, sphere

MSC 2010: 51N20, 52B10, 52B12, 52B15, 65D17

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On Algebraic Minimal Surfaces

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Minimal surfaces have attracted mathematicians' interest for more than two centuries. The intensive research has lead to many results. It is known that the catenoid is the only minimal surface of revolution. The helicoid is the only minimal ruled surface, but not the only helical minimal surface. Scherk's surface is a translational minimal surface. The helicoid is periodic like any helical surface. Doubly and triply periodic minimal surfaces and minimal surfaces of various genus are known and studied. All these have in common that they are not algebraic. These surfaces can be parametrized by means of trigonometric, hyperbolic, and transcendental functions. Some minimal surfaces like Costa's surface can only be parametrized by means of elliptic functions.

Only little is known about algebraic minimal surfaces. Though there are results that go back to Enneper, Geiser, Richmond, Lie, and some others, algebraic minimal surfaces have not gained that much interest. This may be due to the fact that these surfaces are of relatively high degree and that their parametrizations may at first glance be somewhat nasty. However, in some cases it is possible to switch to even polynomial parametrizations.

In this contribution, we shall compare different representations and generations of algebraic minimal surfaces. Beside Weierstraß's representation of minimal surfaces, whether free of integrals or not, even the appropriate use of Björling's formula for minimal surfaces on scrolls turns out to yield algebraic minimal surfaces. An old result by Lie dealing with the evolute of space curves can also serve as a starting point for the construction of rationally parametrized minimal surfaces. The simple parametrizations of these minimal surfaces allow us to study these surfaces in more detail

Key words: minimal surface, algebraic surface, rational parametrization, polynomial parametrization, meromorphic function, isotropic curve, Björling's formula, evolute

MSC 2010: 53A10, 53A99, 53C42, 49Q05, 14J26, 14Mxx.

Minding Isometries of $B-$scrolls in Minkowski Space

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Ruled surfaces in 3−dimensional Minkowski space $\mathbb{R}^3_1$ are surfaces that admit parametrization of the form

$$f(u,v) = c(u) + ve(u), u \in I \subset \mathbf{R}, v \in \mathbf{R}, \tag{2}$$

where c is the base curve and the $e(u)$ is a non-vanishing vector field along c which generates the rulings. Ruled surfaces in $\mathbb{R}^3_1$ are classified with respect to the causal character of their base curve and their rulings (spacelike, timelike and null (lightlike, isotropic)). Among surfaces with null rulings, so called class M_0, $B-$scrolls of null Frenet curves are of special interest. In this work we study local isometries of ruled surfaces that preserve rulings, so called the Minding isometries. We investigate conditions on invariants of $B-$scrolls to obtain such isometries and show that if two $B-$scrolls are locally isometric, then the local isometry preserves their rulings, unless they are $B-$scrolls with constant Gaussian curvature.

Key words: Minkowski space, isometry, ruled surface, B-scroll

MSC 2010: 53A35

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Polygons and Iteratively Regularizing Affine Transformations

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We start with a generic planar n-gon Q_0 with vertices $q_{j,0}$ $(j = 0, \ldots, n-1)$ and fixed reals $u, v, w \in \mathbb{R}$ with $u + v + w = 1$. We iteratively define n-gons Q_k of generation $k \in \mathbb{N}$ with vertices $q_{j,k}$ $(j = 0, \ldots, n-1)$ via $q_{j,k} := u\, q_{j,k-1} + v\, q_{j+1,k-1} + w\, q_{j+2,k-1}$. We are able to show that this affine iteration process for general input data generally regularizes the polygons in the following sense: There is a series of affine mappings β_k such that the sums Δ_k of the squared distances between the vertices of $\beta_k(Q_k)$ and the respective vertices of a given regular prototype polygon P form a null series for $k \longrightarrow \infty$.

The Figure shows a typical example for $n = 7$ with starting polygon Q_0 and $(u, v, w) = (0.5, -0.2, 0.7)$. The 10^{th} generation Q_{10} visually approaches the affine image of a regular star 7-gon.

Key words: affine iterations, affine regularization, regular n-gons

MSC 2010: 51N10, 51N20

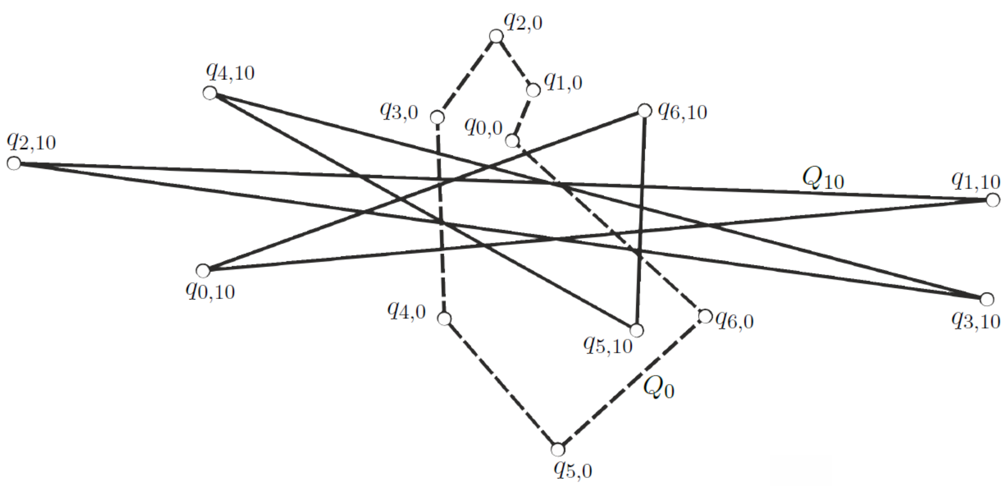

Figure 1: An example for $n = 7$ with $(u, v, w) = (0.5, -0.2, 0.7)$.

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Rational Ruled Surfaces as Trajectories of Minimal Degree Rational Motions

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In the recent paper [1] we proved that there is a unique rational motion of minimal degree in the dual quaternion model of rigid body displacements with a prescribed rational point trajectory. While existence of such motions is trivial, the uniqueness is surprising.

In our presentation, we present results pertaining to uniqueness of the minimal degree motion to prescribed line trajectories (rational ruled surfaces). Here, the situation is quite different:

- A rational motion with the given line trajectory exists if and only if the ruled surface's spherical image is a rational curve.
- In this case, the rational motion of minimal degree is unique if and only if ruled surface and spherical image are of the same degree.

The proof requires a closer investigation of the "generalized steoreographic map" introduced in [2] and the discussion of unique solubility of a system of linear congruences in a polynomial ring. Similar results for plane trajectories (rational torses) can also be derived.

Key words: rational ruled surface, rational motion, dual quaternion, polynomial ring, rational torse

MSC 2010: 70B10, 14J26, 13F20, 65D17

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Auto-Isogonal Cubics

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There are two families of cubic curves which remain invariant under the isogonal transformation with respect to a triangle ABC. One family consists of so-called pivot curves. They are the locus of corresponding points being collinear with a given fixed point, the pivot. In general, these cubics are irrational. The other family consists of 4 pencils of strophoids, which are circular cubics with a node and two orthogonal node tangents. Any two strophoids taken from two different pencils meet in ABC, in the absolute circle-points, and in a pair of corresponding points together with two well defined complex conjugate points. This holds also if two vertices of the given triangle ABC are complex conjugate.

Life and Work of Great Geometer Donald Coxeter

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In this article there are some facts about life and work of a great 20th century geometer Harold Scott MacDonald Coxeter, born in London in 1907, died in Toronto in 2003. It could be said his life was devoted to geometry.

Donald Coxeter had an interesting childhood. As a very imaginative boy he invented his own language *Amabelian*, he played piano and wrote songs.

Coxeter studied at Trinity College, Cambridge, and obtained his PhD in 1931. In 1932 went to Princeton University as Rockefeller Fellow. At Trinity he attended Ludwig Witgenstein's seminars on the philosophy of mathematics.

Greater part of his life he lived in Toronto, Canada where he was professor at Toronto University from 1936.

His main interest was in geometry, where he contributed in the theory of polytopes, non-euclidean geometry, group theory and combinatorics.

He wrote 12 books and 167 articles. Among the most important are: *The real projective plane*, *Introduction to Geometry*, *Regular polytopes*, *Non-euclidean geometry, geometry revisited* (with S. L. Greitzer).

Coxeter and famous painter M. Escher inspired each other in their work. They especially discussed Escher's *Circle Limit* series based on hyperbolic tessalations.

Coxeter was a Fellow of Royal Society, Honorary Fellow of the Edinburgh Math Society, and he was awarded with Royal Society Sylver Medal.

The Canadian mathematical Society had awarded Coxeter James Prize in Coxeter's honour.

He attended conference in Budapest in 2002.

Key words: geometry, polyhedra

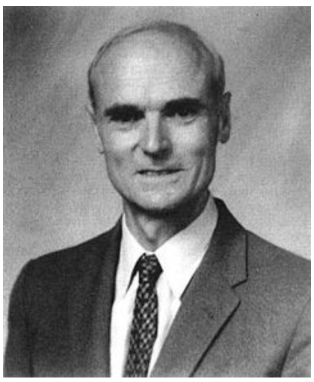

Figure 1: Harold Scott MacDonald Coxeter (9 February 1907 – 31 March 2003)

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On Brocard Points of Harmonic Quadrangle in $I_2(\mathbb{R})$

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In this talk we present several results concerning the geometry of a harmonic quadrangle in the isotropic plane $I_2(\mathbb{R})$. We consider the standard cyclic quadrangle with the circumscribed circle given by $y = x^2$ and the vertices chosen to be $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$, and $D = (d, d^2)$, with a, b, c, d being mutually different real numbers, $a < b < c < d$. The *harmonic quadrangle* in the isotropic plane is a standard cyclic quadrangle with a special property: vertices A, B, C, and D are chosen in a way that tangents $\mathcal{A}$ and $\mathcal{C}$ at the vertices A and C, respectively, intersect in the point incident with BD, and tangents $\mathcal{B}$ and $\mathcal{D}$ at the vertices B and D, respectively, are intersected in the point incident with AC.

We show that there exist a unique point P_1, so-called the first *Brocard point*, such that the lines P_1A, P_1B, P_1C, and P_1D form equal angles with the sides AB, BC, CD and DA, respectively. Similarly, the second Brocard point is defined as the point such that the lines P_2A, P_2B, P_2C, and P_2D form equal angles with the sides AD, DC, CB, and BA, respectively.
Figure 1 illustrates two Brocard points P_1, P_2, the diagonal point $U = AC \cap BD$, and midpoints M_{AC}, M_{BD} of the sides AC, BD, respectively, lying on a circle.

We compare the obtained results with their Euclidean counterparts.

Key words: isotropic plane, harmonic quadrangle, Brocard points

MSC 2010: 51N25

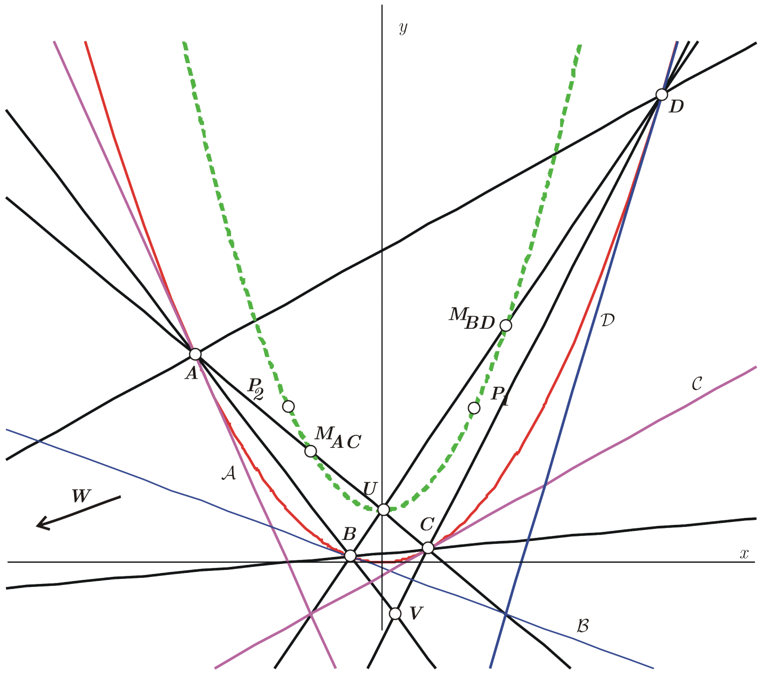

Figure 1: Brocard points P_1, P_2 of the harmonic quadrangle $ABCD$.

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Examples of Convex Polytopes with Odd Translative Kissing Numbers

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The translative kissing number $H(K)$ of a d-dimensional convex body K is the maximum number of mutually nonoverlapping translates of K that can touch K. The translative kissing number of a convex body is often called the Hadwiger number of the convex body as well.

In dimension 2, $H(K)$ can be 6 or 8, that is, $H(K)$ is always an even number for a convex disc K.

In dimensions 3 and higher, the situation is very different: During previous years, some convex bodies were found in dimension d, for every $d \geq 3$, whose translative kissing numbers are odd (see [1]). However, none of those examples are polytopes, since they contain some strictly convex arcs on their boundaries.

Now, we present families of explicit examples of convex polytopes in every dimension $d \geq 3$ whose translative kissing numbers are odd.

Key words: translative kissing number, Hadwiger number, packing, convex polytope

MSC 2010: 52C17

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Minkowski Free-Form Laces

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This paper brings basic information about possible extension of the concepts of Minkowski sum and product and Minkowski combinations of manifolds to modeling free-form curve segments and surface patches. Some of intrinsic geometric properties of surface patches generated by means of this method are presented, with illustrations of particularly interesting resulting geometric forms. Presented generating principles for modelling curve segments and surface patches are discussed as tools for modelling of geometric objects, differential manifolds in the Euclidean space of an arbitrary dimension. Let us restrict considerations specifically to free-form curve segments in $\mathbf{E}^3$ represented parametrically by uniform vector maps on the unit real interval $I = \langle 0, 1\rangle \subset \mathbf{R}$

$$\mathbf{K} : \mathbf{k}(u) = \sum_{i=1}^{n} A_i Pl_i(u) = \left(\sum_{i=1}^{n} xa_i Pl_i(u), \sum_{i=1}^{n} ya_i Pl_i(u), \sum_{i=1}^{n} za_i Pl_i(u) \right), u \in I,$$

$$\mathbf{L} : \mathbf{l}(v) = \sum_{i=1}^{n} B_i Pl_i(v) = \left(\sum_{i=1}^{n} xb_i Pl_i(v), \sum_{i=1}^{n} yb_i Pl_i(v), \sum_{i=1}^{n} zb_i Pl_i(v) \right), v \in I,$$

where $A_i = (xa_i, ya_i, za_i), B_i = (xb_i, yb_i, zb_i)$ are vertices of the curve basic determining polygons and Pl_i are interpolation polynomials in the respective form. Minkowski summative combination of the two curves is a family of translation surface patches defined on unit square $I^2 \subset \mathbf{R}^2$ as

$$k \cdot \mathbf{K} \oplus \mathbf{l} \cdot \mathbf{L} : \mathbf{s}(u, v) = k \cdot \mathbf{k}(u) + l \cdot \mathbf{l}(u) = k \cdot \sum_{i=1}^{n} A_i Pl_i(u) + l \cdot \sum_{i=1}^{n} B_i Pl_i(v)$$

that can be generated by translation of one curve segment along the other one. Their differential characteristics are represented by derivatives of the curve vector maps.

Minkowski partial sum and partial summative combinations of curves $\mathbf{K}$ and $\mathbf{L}$ are similarly defined for the two free-form curve segments determined on $I \subset \mathbf{R}$ by the same parameter t. Resulting family of curve segments in $\mathbf{E}^n$ represented by vector maps $\mathbf{S} : \mathbf{s}(t) = k \cdot \mathbf{k}(t) + l \cdot \mathbf{l}(t)$ is two-parametric family of summative laces with special properties. It can be derived as a family of curves determined on basic polygons that are respective Minkowski partial combinations of determining polygons of the operand curves.

Illustration of Minkowski sum of two quadratic Bézier curve segments is presented in Figure 1, summed curves are positioned in two perpendicular planes, while their partial Minkowski sum is illustrated on the right, too. Minkowski partial sum of two planar cubic Bézier curve segments is displayed in Figure 2.

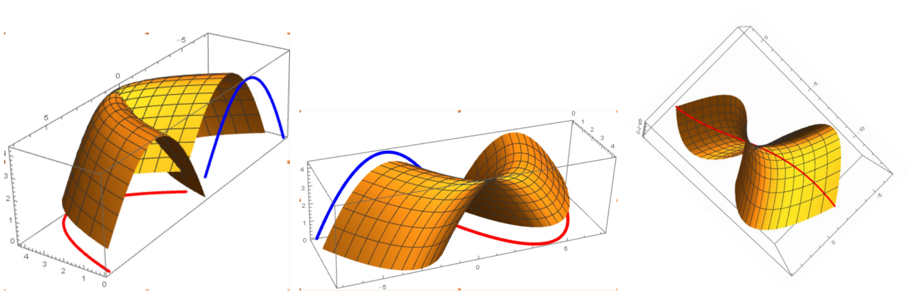

Figure 1: Minkowski sum and partial sum of two Bézier curve segments

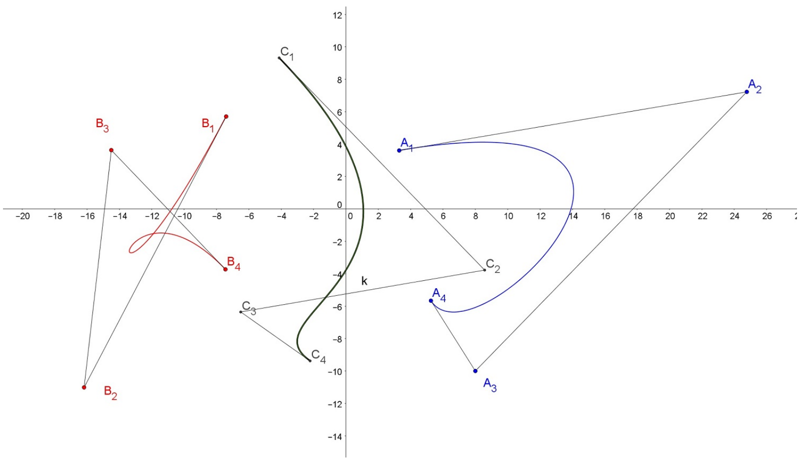

Figure 2: Minkowski partial combination of two Bézier curve segments

Key words: Minkowski sum, Minkowski partial combination, Bézier curve segments

MSC 2010: 65D17, 51H30, 68U07.

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Parallelograms and Equations in Some Special Subclasses of IM-Quasigroups

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IM-quasigroups are quasigroups in which the additional identities of idempotency and mediality hold. The concept of parallelogram can be introduced in any IM-quasigroup, but in some special subclasses it can be defined more elegantly. We study relations between this concept and the solutions of basic equations of quasigroups. We present some results concerning these relations in some special subclasses of IM-quasigroups in order to obtain general conclusions.

Key words: IM-quasigroup, parallelogram, equation

MSC 2010: 20N05

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Special Conics in a Hyperbolic Plane

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In Euclidean geometry we find three types of special conics, which are distinguished with respect to the Euclidean similarity group: circles, parabolas, and equilateral hyperbolae. They have, on one hand, special elementary geometric properties (c.f. [2]) and on the other are strongly connected to the "absolute elliptic involution" in the ideal line of the projective enclosed Euclidean plane. Therefore, in a hyperbolic plane – and similarly in any Cayley Klein plane – the analogue question has to consider as well projective geometric properties as also hyperbolic-elementary geometric properties.

As a first and well-known example we consider circles: In a hyperbolic plane with the (real) "absolute conic" Ω a circle c is a conic touching Ω twice in algebraic sense, i.e., this projective geometric approach gives already three types of hyperbolic circles: (1) proper circles touching Ω in a pair of conjugate complex points, (2) limit circles which hyperosculate Ω, and (3) distance circles touching Ω in a pair of real distinct points. While the Euclidean elementary definition of a circle is the planar set of points having equal distance from a centre point, the analogue in hyperbolic geometry is true only for h-circles of type (1) and can be modified for h-circles of type (3). For h-circles of type (3) the radius length is not finite, a property, which we also find at Euclidean parabolas.

Euclidean circles can be generated via indirect congruent pencils of lines, which expresses the property of constant angle at circumference and especially the property of Thales. In an h-plane two pencils of orthogonal lines generate the so-called "Thales conic" (resp. "Thaloid", as it is called by N. J. Wildberger, see [3]), which is never an h-circle of type (1) and (2). Type (3) h-circles occur if and only if the vertices of the h-orthogonal pencils both are ideal points on Ω.

Parabolas are treated in [1]. The place of action there, the "universal hyperbolic plane", is the full projective plane endowed with an "absolute (regular) hyperbolic polarity". The polarity usually is given by the real conic Ω and rules of orthogonality, h-distance and h-angle measure. Considering h-geometry as a sub-geometry of the projective geometry is F. Klein's point of view. A puristical point of view allows only the inner domain of Ω for being a proper h-plane. In the following we try to have both points of view in mind and will also use F. Klein's model of an h-plane.

The main part of this talk will concern conics derived from properties of the Euclidean equilateral hyperbola following the above presented systematic treatment for circles.

Key words: conic section, hyperbolic plane, Thales conic, equilateral hyperbola

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Inside-out Sphere: University Tram Station

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University tram station design is a part of a course we are taking part in- "Digital Stereotomy Geometry - Processing & Construction" in the Master degree at the Politecnico di Milano. There, we examined stereotomy and construction which has already inflamed the birth of modern descriptive geometry between 16th and 18th centuries, and is again on stage as a catalyst subject in the era of digital computational design. This approach has led us throughout the planning process, starting from digitally re-mastering some vaulted structure from the Philibert De l'Orme's treatise (Figure 1).

Our project starts from analyzing simple geometrical shapes and understanding the logic behind these component structures. Using simple forms allows us a variety of options- by using parts of them in a logical and efficient manner. We used the same logic to the current planning process with using only sphere and its variations as our design challenge.

The project is a small tram station located in front of the Politecnico di Milano University. It should contain people, movement and a convenient space for waiting and welcoming for those who arrive there, mostly students.

In our project we are trying to offer a space that combines open and closed, intimate and shared space, all at the same time. We believe this moment of arriving and leaving the university area is a moment of transition in which you are still inside your own mind but starting your interaction with the outside world. For that reason, we are designing a shelter that protects on the one hand, but is delicate and light on the other.

From a pure sphere, we developed a structure made of spirals. The structure is composed by spiral surfaces in 2 different directions. This system creates a stable and solid structure that allowed us to have the easy sense we aimed for, as well (Figure 2).

In addition, in our design we had an important goal of taking in consideration the environmental conditions such as the trees, that are decorating the avenue, and making them a basis for our design. For making our design suitable to the volume of the trees, the high of people, the entrance of the tram, the cars and many other physical limitations, we used the sphere shape and its variations only.

Key words: descriptive geometry, architectural geometry, sphere, spiral, light

MSC 2010: 00A66, 51N05, 01A05, 97U99

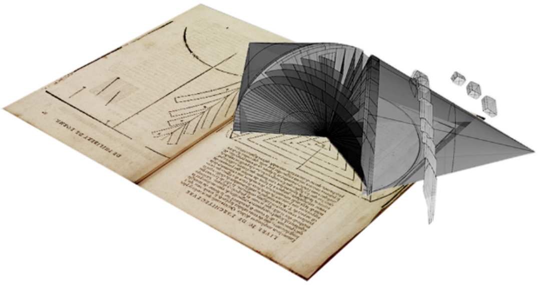

Figure 1.

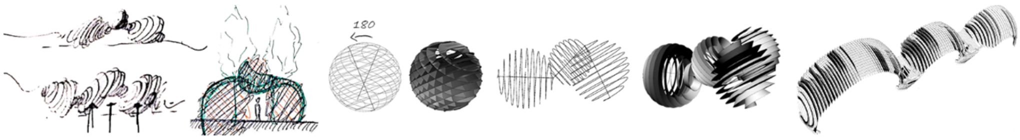

Figure 2.

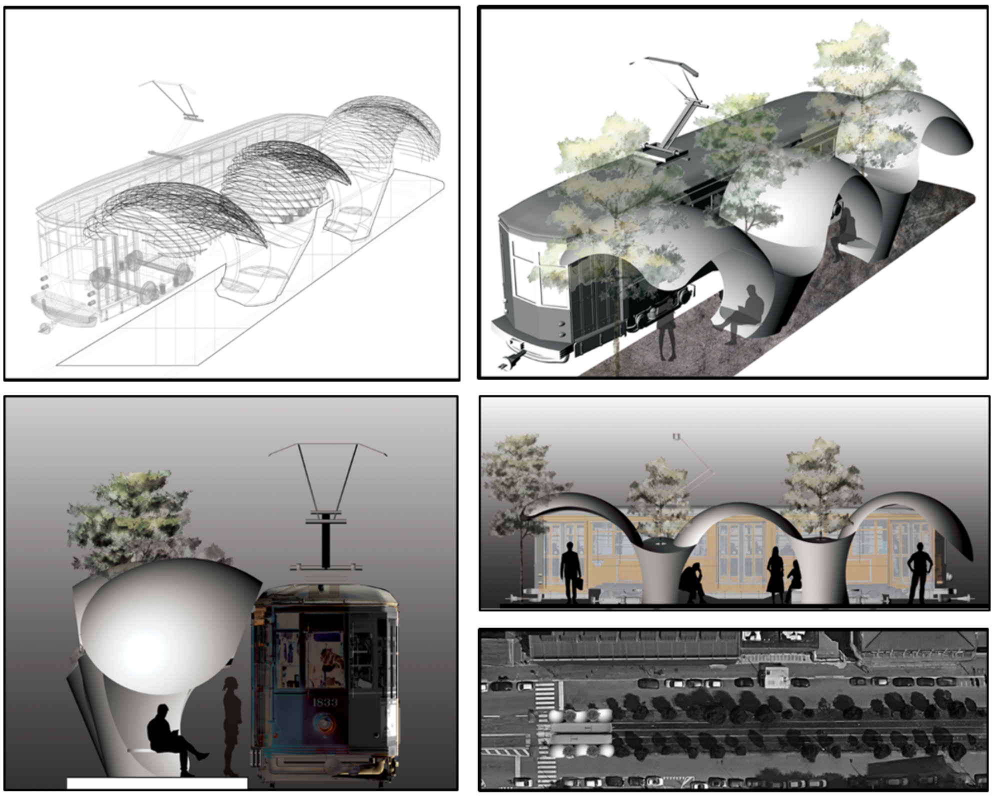

Figure 3: Wireframe and rendered views of the project (digital model of the tram by Matteo Cavaglià).

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Posters

Searching for Patterns in the Learning of Mathematics with Help of Learning Analytics: A Tale of Two Cases

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Algorithms, tools and representations used in learning analytics give new insights into the very complex processes of learning and teaching. Our aim is to follow the progress of students learning Mathematics at tertiary level, with emphasis on the complex-problem solving tasks with assessment of standard math skills and theoretical knowledge.

The poster will present two case studies covering courses taught at the Faculty of Organization and Informatics, University of Zagreb: Selected Chapters of Mathematics taught at the undergraduate level of Information Systems and Discrete Mathematics with Graph Theory taught at the master level of Informatics.

We are interested in patterns occurring in the way students progress through the course. All categories of students with their typical behaviour are of interest. We are especially interested in the often overlooked "middle". Interesting and unexpected findings will be highlighted.

Special emphasis will be put on the investigation of relationship between achievement on complex problem solving tasks and overall success in standard mathematical tasks.

Transitions and trajectories of students through various assignments of the course will be illustrated with the help of interactive Sankey (ribbon) diagrams from the start to the end of the course.

Key words: mathematical education, learning analytic, geometrical modelling

MSC 2010: 91E45, 97B10

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Circular Curves of the 3rd Class in the Quasi-Hyperbolic Plane Obtained by a Projective Mapping

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The metric in the quasi-hyperbolic plane is induced by an absolute figure $\mathcal{F}_{\mathbb{QH}} = \{F, f_1, f_2\}$, consisting of two real lines f_1 and f_2 incident with the real point F. A curve of class n is circular in the quasi-hyperbolic plane if it contains at least one absolute line.
The curves of the 3rd class can be obtained by projective mapping, i.e., obtained by projectively linked pencil of curves of the 2nd class and range of points. The circular curves of the 3rd class of all types, depending on their position to the absolute figure, can be constructed with projectively linked pencils.

Key words: projectivity, circular curve of the 3rd class, quasi-hyperbolic plane

MSC 2010: 51A05, 51M15, 51N25

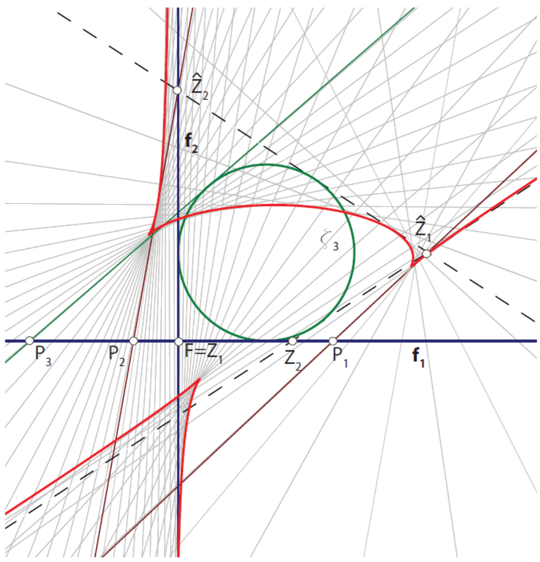

Figure 1: Circular curve of the 3rd class of type (2,1) in the quasi-hyperbolic plane.

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Sensitivity Analysis Based on Data Visualization

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Sensitivity analysis is the procedure where one analyzes to what extent do possible value changes of the mathematical model input variables affect this model's output values. Likewise, based on the sensitivity analysis, one can estimate the different scenario consequences of some decision realization. The procedure of sensitivity analysis is determined by the mathematical model, based on which the decision is made. From the properties of this model, the possibility to answer a certain amount of questions arises, by which the decision-maker can with greater certainty predict the consequences of his decision. On this poster, we shall represent how the visualization of the sensitivity analysis results can contribute to better understanding and interpretation of the same.

Key words: sensitivity analysis, visualization

MSC 2010: 49Q12

Geometric Compositions in Art Installations

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Art compositions are often creating pattern compositions using multiplication of a single geometric element. Contemporary art uses polyhedrons – Platonic solids as well as different irregular polyhedral faces in creating art installations [1].

Two tendencies are essential for ancient mathematics, Pythagorean and Euclidean geometry. Pythagorean geometry is seen as symbolic system of elements and relations, i.e., geometric shapes and their spatial relations are perceived as numeric or visual representations of the divine and natural phenomena. Euclidean geometry is seen as a formal system of displaying space, spatial forms and their relations [2].

This research seeks to find and explain common geometric and visual principles and to give visual coherence to design through visual structure in contemporary art as well as in art produced in geometry educational process. Specific attention is on development of geometrical compositions from Platonic solids to complex geometrical forms in different artistic approaches. The research analyses two art installations.

The first art installation, hexahedron composition is a product of students' research study of geometry of multiplication of the same geometric solid (cube), regular hexahedron. All elements are of the same size arranged to create depth of space shifted in direction of one axis, emphasizing the 3D effect. The art installation is produced during 2016 on Geometry workshop by the group of students of Landscape architecture and was the part of annual exhibition in June 2016 (Figure 1).

The second art installation is an abstract geometrical composition of geometrically different elements. In the basis of the each compositional element (made of composite material – plastic and paper) is rectangle. Folding surfaces composition is a product of artist's vision of abstract composition of open folded polygonal surfaces. Folding surfaces are consisting of different irregular polyhedral faces. Art installation "Light and Day" by artist Mira Ranković is a part of project funded by Ministry of Culture in 2015, realized in 2016 and is a part of permanent exhibition of Faculty of Forestry.

Key words: geometry, art, art instalation, geometry workshop

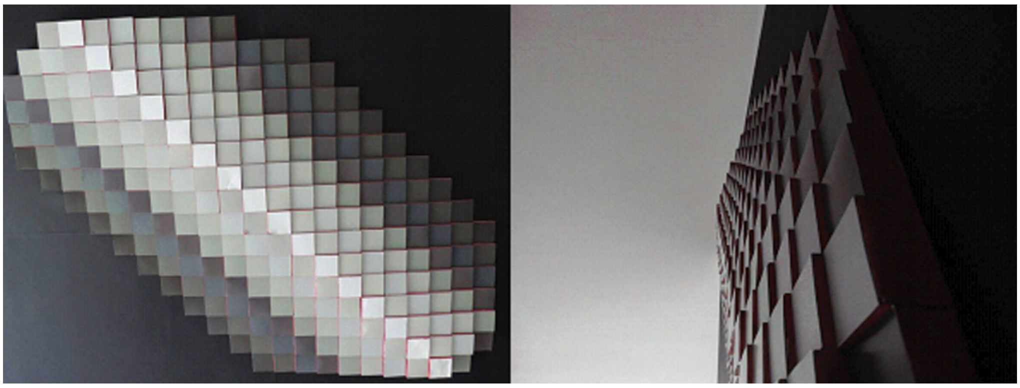

Figure 1: Hexahedron composition; Art installation by group of students on Geometry workshop in 2016.

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