4th Croatian Conference on Geometry and Graphics
Vodnjan (Peroj), September 2 – 6, 2018

ABSTRACTS

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PUBLISHER:
Croatian Society for Geometry and Graphics

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Supported by the Ministry of Science and Education of the Republic of Croatia and the Foundation of Croatian Academy of Sciences and Arts.

Contents

Plenary lectures

The Persistent Homology Pipeline: Shapes, Computations, and Applications

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The theory of persistent homology provides a multi-scale summary of homological features which is stable with respect to noise [1, 2, 3]. These properties make homological algebra applicable to a growing range of application areas (in geometry and beyond) and give rise to the field of topological data analysis. This success of linking theory and applications has posed the challenge of computing persistence on large data sets. Typical questions in this context are: How can we efficiently build combinatorial cell complexes out of point cloud data? How can we compute the persistence summaries of very large cell complexes in a scalable way? Finally, how does the computed summary lead us to new insights into the considered application?

In my talk, I will introduce the field of topological data analysis in detail and discuss challenges, application areas and recent developments.

Key words: topological data analysis, persistent homology, applied topology

MSC 2010: 55U99, 68W25, 52C45

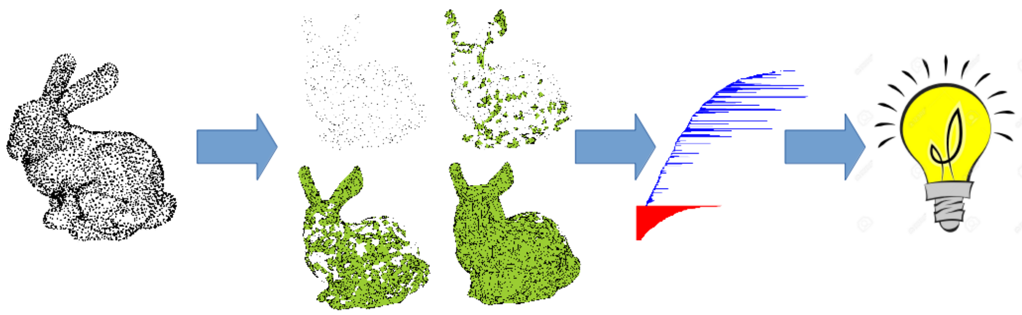

Figure 1.

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Experiments with Soap Films and Surfaces with Constant Mean Curvature

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We consider the study of the geometry of constant mean curvature surfaces in Euclidean space. We will prepare a variety of experiments with soap films and soap bubbles and we will review some results in the theory and their proofs. Finally we will pose some open problems.

Key words: mean curvature, soap film, soap bubble

MSC 2010: 53A10

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Incomputable yet Physically Tangible Numbers

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Do you like music? There are mysterious numbers which are responsible for the way many of our favorite instruments sound called *eigenvalues*. They can be thought of as relatives of the mysterious number, π. When we collect all the eigenvalues, this collection of numbers is called *the spectrum*. These numbers determine many important physical phenomena including: the sounds caused by vibrations, the way waves travel, the flow of heat, and the energy of quantum particles. Important as they are, in general, we are unable to compute them exactly. Nonetheless, mathematicians and mathematical physicists are able to glean important information about the spectrum using a variety of techniques. Here, I will focus on connections between geometry and the spectrum. As an example, we will see that the spectrum detects geometric symmetry.

Geometric Constructions of Polyhedra

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We present several geometric constructions by which 3-dimensional polyhedra can be created from planar figures or from simpler polyhedra. This method is used (among others) in some courses at Ybl Faculty of Szent István University to develop spatial abilites of students while they are working in a virtual 3D environment. This also helps the students to have a better understanding of both combinatorial and metric properties of polyhedra. Considering symmetries and applying appropriate geometric transformations play important roles in these constructions.

The basic construction steps in the 3-dimensional space are mostly analogous to the planar compass and straight edge basic construction steps, and most 3D geometry software offer those as construction tools. Depending on software, some more advanced construction steps may be used (those typically simplify the construction), like slicing a polyhedron with a plane, or creating the convex hull of the union of several polyhedra.

To model polyhedra, the software programs AutoCAD, Cabri 3D, and GeoGebra were used in the courses 'Space Geometry with Computers' and 'Mathematics and Geometry in Architecture' at Ybl Faculty of Szent István University during the previous seven academic years. The course topics regarding polyhedra were modified several times: Beside Archimedean solids, the other major class of polyhedra in focus was changed from time to time.

Specific types of polyhedra which were constructed during these courses:

- Archimedean solids: Each face is a regular polygon, and the polyhedron is convex and vertex-transitive (excluding regular solids, prisms, and antiprisms).
- Uniform star polyhedra: Each face is a regular polygon (convex polygon or star polygon), the polyhedron is vertex-transitive, and self-intersecting.
- Johnson solids: Each face is a regular polygon, the polyhedron is convex, but it is not vertex-transitive.
- Catalan solids: Duals of Archimedean solids, that is, face-transitive convex polyhedra having regular vertex figures.
- Nonconvex twisted prisms.
- Geodesic polyhedra: Each face is a triangle, the vertices lie on a sphere, the polyhedron approximates a sphere better as the number of vertices increases, and the edges approximate arcs of great circles of a sphere, [1].
- Goldberg polyhedra: Duals of geodesic polyhedra of rotational icosahedral symmetry, having only pentagonal and hexagonal faces, [2].

- Polyhedra of a cell system of convex cells representing the face lattice of a 4-dimensional regular solid, [3].

We discuss the geometric constructions for every type of polyhedra created during the courses and their difficulties for the students.

Key words: polyhedron, symmetry, geometric transformation

MSC 2010: 52B10, 52B15, 97G40

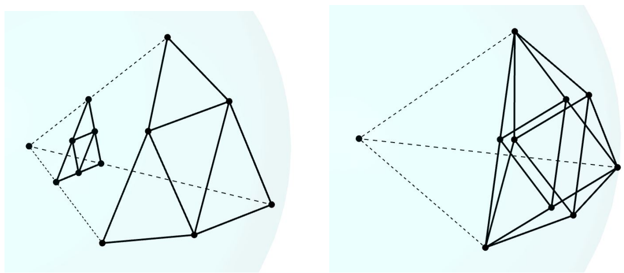

Figure 1: Projecting the triangles of a subdivided triangular face radially to the boundary of a sphere when the vertices of the face lie inside, and on the boundary of the sphere, resp.

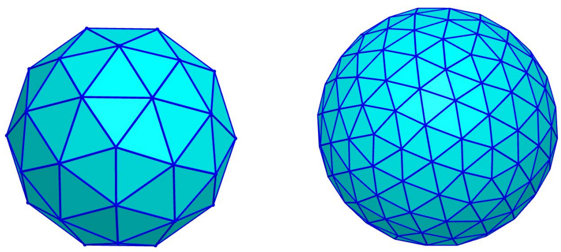

Figure 2: Geodesic icosahedra L2 and L3, created by subdivision of triangular faces and the spherical projection of the obtained new triangles in the case of an icosahedron, applying this procedure once, and twice, resp.

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Contributed talks

3D Static Equivalency Using Grassmann Algebra

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When a system of external forces is acting on a body, it is easier to understand their overall effect on the body if they are replaced by a simpler system having the same external effect. Two systems of forces are statically equivalent if their contribution to the conditions of static equilibrium is the same.

According to the principle of transmissibility, a force can be shifted along its line of action. Therefore, the force is considered as a line-bound vector with Plücker coordinates $\mathbf{F} = (f_{01}, f_{02}, f_{03}, f_{23}, f_{31}, f_{12})$. The first three coordinates represent vector $\mathbf{f}$ of the force $\mathbf{F}$, and the second three represent moment vector $\mathbf{m}$ of $\mathbf{f}$ about the origin of the coordinate system.

To perform and visualize procedures of replacing one system of forces with statically equivalent one, we are developing a computer program based on algebraic translations of descriptive geometry operations.

Key words: 3D graphic statics, static equivalency, line geometry, Grassmann algebra

MSC 2010: 70C20, 68U05, 65D18

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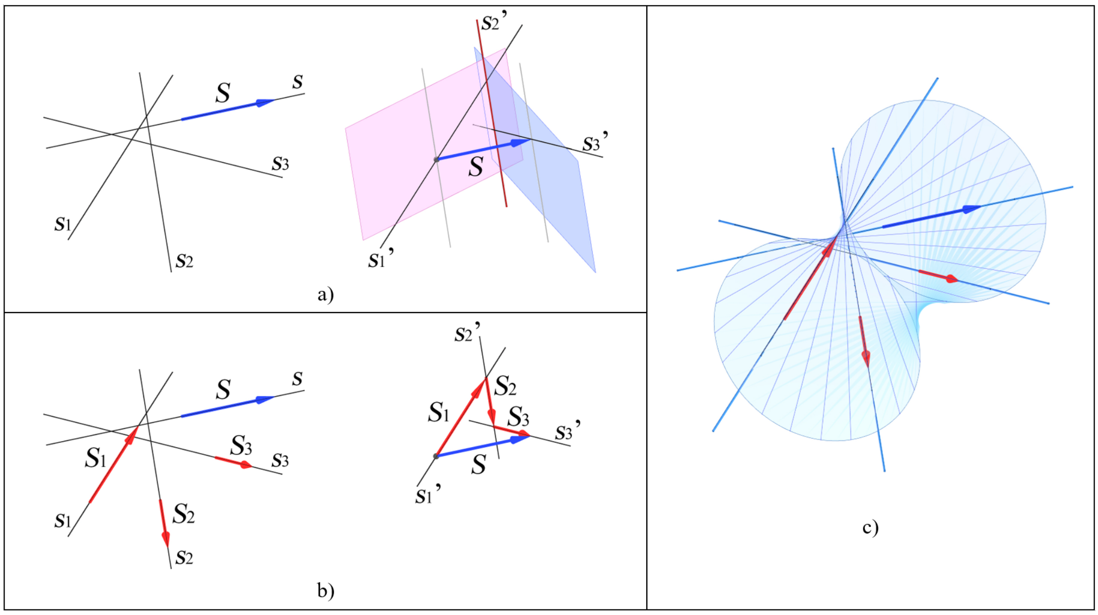

Figure 1: Replacing a single force with three forces acting along generators of a regulus

Envelopes of Systems of Convex Curves and their Applications

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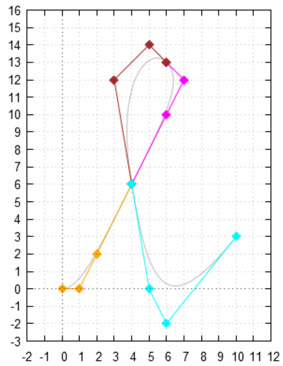
Envelope theory is a classical topic of differential geometry and its results are applied in the field of robotics, design, optics and other areas.

For some curve types (e.g. line segments, lines, conic sections) and some special types of one-parametric systems of sets, the envelopes are known. However, the general task leads to differential equations.

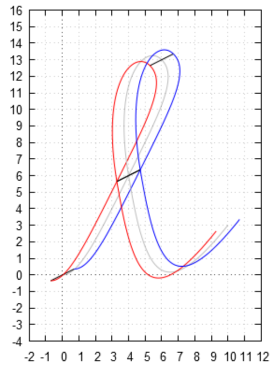
In our work, we present the case of line segments and simple conic sections (see fig. 1) subjected to arbitrary affine change of shape. We propose a set of algorithms to approximate the envelope of a system of simple convex sets in the Euclidean plane bounded by piecewise smooth curves. We mention the properties of the envelope that allow it to be represented accurately by means of rational splines, if that is the case.

We use differential-geometric or algebro-geometric (if possible) properties of the one-parametric system and we estimate the properties of the envelope.

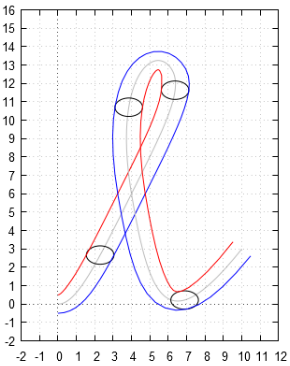
We mention applications of our results in several sub-areas of visual art and industrial design, e.g. in offset determination, tolerance area of robotic movements, font design etc.



(a) Original piecewise cubic Bézier spline curve.



(b) Envelope of a line segment (black).



(c) Envelope of an axial ellipse (black).

Figure 1: Envelopes (red and blue curves in (b) and (c)) of systems of curves corresponding to given piecewise cubic Bézier spline curve (gray in (a)).

Key words: differential geometry of envelopes, convex sets, spline approximations of envelopes

MSC 2010: 65D17, 65D07, 41A15, 53A04

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Teaching Mathematics – a Task that Requires Continuous Adaptation

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Teaching math at any of the engineering faculties is a task that requires continuous adaptation. On the one hand, we are obliged to follow the needs of our fellow engineers who teach professional subjects; on the other, we have to adapt to the changes related to new types of knowledge, learning styles, ways of communication, and attitudes of our college students, especially freshmen.

It is a part of our professional motto to handle both of these challenges with a lot of enthusiasm. We do it on a daily basis as individuals; however, the effect is greater when performed on faculty/university level or in even wider range.

The European Society for Engineering Education (SEFI), in particular its Mathematics Working Group (MWG) [5], is the institution that one can rely on in this respect.

The goal of SEFI's MWG is to provide a discussion forum and orientation for those who are interested in the mathematical education of engineering students in Europe. A contribution to this goal is the group's curriculum document which was first issued in 1992. In 2002, a second edition was published which brought the document more in line with current curriculum practices by formulating ... a list of concrete content-related learning outcomes. ... The intention of the third edition (2013) is to state, explain and exemplify a framework for systematically including such higher-level learning goals based on state-of-the-art educatorial research. For this purpose, the ***competence concept*** *... is used (Alpert, B. et al. 2013, p.7).*

The concept of mathematical competence (MC) was developed in Denmark within the scope of Danish KOM project (KOM: Competencies and the Learning of Mathematics), in order to create a platform for a profound reform of Danish mathematics education at all educational levels [6, 7].

Mogens Niss (Niss 2003, p. 6/7), the director of the KOM project describes mathematical competence as ... *the ability to understand, judge, do, and use mathematics in a variety of intra- and extra- mathematical contexts and situations in which mathematics plays or could play a role. Necessary, but certainly not sufficient, prerequisites for mathematical competence are lots of factual knowledge and technical skills ...*

Within the results of the KOM project, published in 2011, apart from the definition of mathematical competence itself, a list of eight mathematical competences being its part is described. These are: thinking mathematically; posing and solving mathematical problems; modeling mathematically; reasoning mathematically; representing mathematical entities; handling mathematical symbols and formalism; communicating in, with, and about mathematics; and making use of aids and tools.

The three SEFI's MWG documents ([2], [3], [1]) have been the guidelines to me throughout my teaching experience. I felt encouraged by these documents to introduce visualisations and animations in my lessons, to emphasize the purpose of learning math units, to highlight the learning outcomes and connect them with the lectured topics as well as with the needs within engineering courses. But, when trying to awaken awareness among my students of the need for MC and stimulate their use in engineering context, I found it difficult to carry it out in the first years of study, because applying MC in solving real-life problems requires students' active involvement and some already mastered engineering knowledge.

It turned out that the above mentioned can be fully achieved in math courses taught at higher years of study. Such courses are usually taken by less than 50 students, which makes active and team-based student work possible.

One of such courses is Discrete Mathematics, an elective course taught within the undergraduate programme in geodesy and geoinformatics. Students attending the course receive or choose on their own one real-life project task to be solved and presented. In doing so, they are encouraged to apply the adopted mathematical competencies. One of such works will be presented at the Conference.

Key words: mathematics education, mathematical competence, teaching-learning processes

MSC 2010: 86A30, 90C35, 97B10, 97C70

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Playing with the Constructions of Limaçon

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The Limaçon of Pascal is a well studied curve of the fourth order. We know a lot about its properties and a number of ways to construct it. For instance, Limaçon is a pedal curve of a circle, an inverse of a conic when the center of the ordinary circle inversion is the focus of the conic, or an envelope of circles through a given point with the centers on a given circle. Furthermore, due to different ways of construction, the Limaçon belongs to several types of curves or is a special type of some other plane curve. It can be constructed as a conchoid of a circle or it can be defined as an epicycloid, i.e., a roulette formed by the path of a point fixed to a circle when that circle rolls around a circle.
In this presentation we will talk about curves obtained by constructions similar to those of the Limaçon, but changing some basic elements of the constructions. For an example, let say that, in the construction of Limaçon as an envelope of circles, we take that the centers of those circles lie on a given conic and not on a circle.

Key words: Limaçon, conic, roulette, conchoid

MSC 2010: 14H50, 51N25

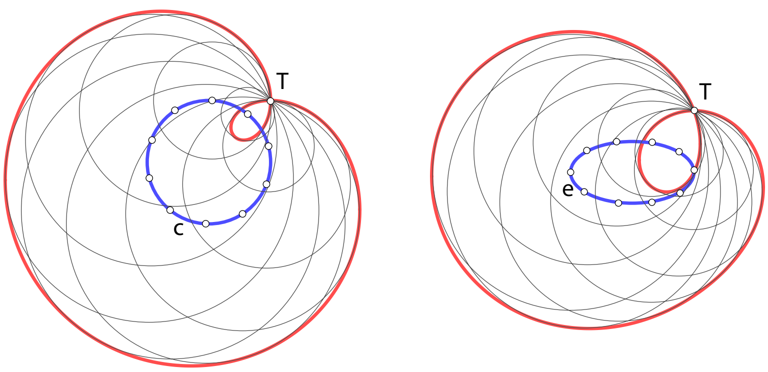

Figure 1: Envelope of circles through a given point T with the centers on a given circle c (left) or on a given ellipse e (right)

Isoptic Curves of Cycloids

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The history of the isoptic curves goes back to the 19th century, but nowadays the topic is experiencing a renaissance, providing numerous new results.

In this talk, first, we define the notion of isoptic curve and outline some of the well-known results for strictly convex, closed curves. The formulas of the isoptic curves to conic sections are well known since 1837.

Overviewing the types of centered trochoids, we will be able to give the parametric equations of the isoptic curves of hypocycloids and epicycloids. Furthermore, we will determine the corresponding class of curves.

Simultaneously, we show that a generalized support function can be given to these types of curves in order to apply and extend the results for strictly convex, closed curves.

Key words: isoptic curves, hypocycloids, epicycloids, centered trochoids

MSC 2010: 51N20, 51L10

Directed Packings of Circles in the Plane

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Let r_0 be non-negative and $(r_n)_{n>0}$ be a sequence of positive real numbers. We define a sequence $K_n = (S_n, 1/r_n)$ of circles in the plane so that K_n is centered at S_n, has the curvature r_n, the circles K_0 and K_1 touch externally at the origin, and K_n touches externally both K_{n-1} and K_{n-2} for $n \geq 2$. Out of two possible positions for S_n we always choose the one farther from the origin. We investigate how the behavior of the sequence (S_n) of circle centers depends on the properties of the sequence (r_n) of their curvatures.

Killing Magnetic Curves in Sol Space

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Magnetic curves represent trajectories of charged particles moving on a Riemannian manifold under the action of a magnetic field. The study of magnetic curves in arbitrary Riemannian manifolds was developed in the early 1990's. Interesting results on magnetic curves in Euclidean space [7], Sol space [5], Sasakian manifolds [3] and cosymplectic manifolds [4] appeared recently.

A vector field X is a *Killing vector field* if the Lie derivative with respect to X of the ambient space metric g vanishes. One can say that Killing vector fields are the infinitesimal generators of isometries.

The trajectories corresponding to the Killing magnetic fields are called the *Killing magnetic curves*. Killing magnetic curves in Euclidean space $\mathbb{E}^3$, Minkowski spacetime $\mathbb{E}^3_1$ and in $\mathbb{S}^2 \times \mathbb{R}$ space were studied in [1, 2, 6] respectively.

In this talk we study the Killing magnetic curves in the 3-dim Sol space.

Key words: Magnetic curve, Killing vector field, Sol space

MSC 2010: 53C80, 53C30

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Geometry, CAD-implementation and 3D-printing of Gears

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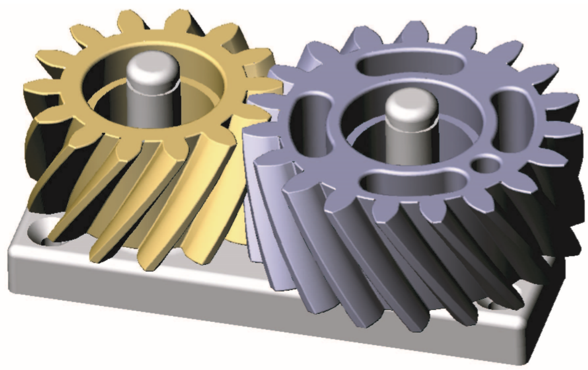
Involute, helical and bevel gears are standard in mechanical engineering applications. The basis for their construction are various geometric and kinematic aspects as circle involutes and developable surfaces. Hence, these applications can serve as a key motivation in the geometry education of mechanical engineers. In the respective courses the students learn that the knowledge of geometric issues is essential. For example, if one wants to correctly implement a helical gear pair in a CAD environment one has to know about parameterizations of circle involutes and properties of tangent surfaces. We present a project which has to be elaborated by mechanical engineering students at TU Graz and MU Leoben. This project contains the implementation of helical gear pairs in a CAD environment and the additive manufacturing of a respective 3D-model by means of an FDM printer. A more advanced topic that we will discuss is the construction of straight and helical bevel gears which is based on spherical circle involutes and geodesics on right cones and their tangent surfaces [1].



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Remarkable Kinematic Mechanisms in the Animal World

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Some kinematic mechanisms in the animal world seem so bizarre to us that they make us think of forgeries or animated cartoon sequences. Only when we look closer and engage intensively with the anatomical idiosyncrasies of individual animals do we realize that the mechanisms are indeed feasible and represent masterpieces of evolution. There is a fascinating story involving the deep-sea shark, which was filmed for the first time in the act of eating prey by a Japanese research team (Fig. 1, middle and right). Another example is the much smaller sling-jaw wrasse, which appropriately bears the species name "insidiator" (Fig. 2). The mechanism that sprains its jaws in a bizarre-looking way will be discussed in great detail. Other examples are the kinematic four-link chain in the monitor lizard head (Fig. 1, left) and the hydraulics in insect wings in order to prepare flight (cockchafer, Fig. 3).

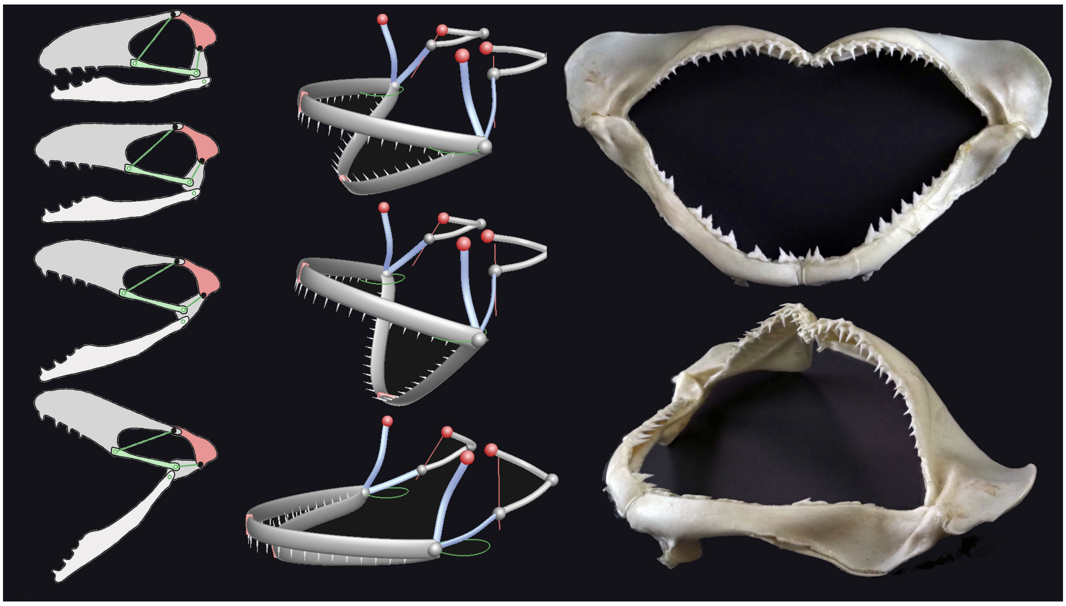

Figure 1.

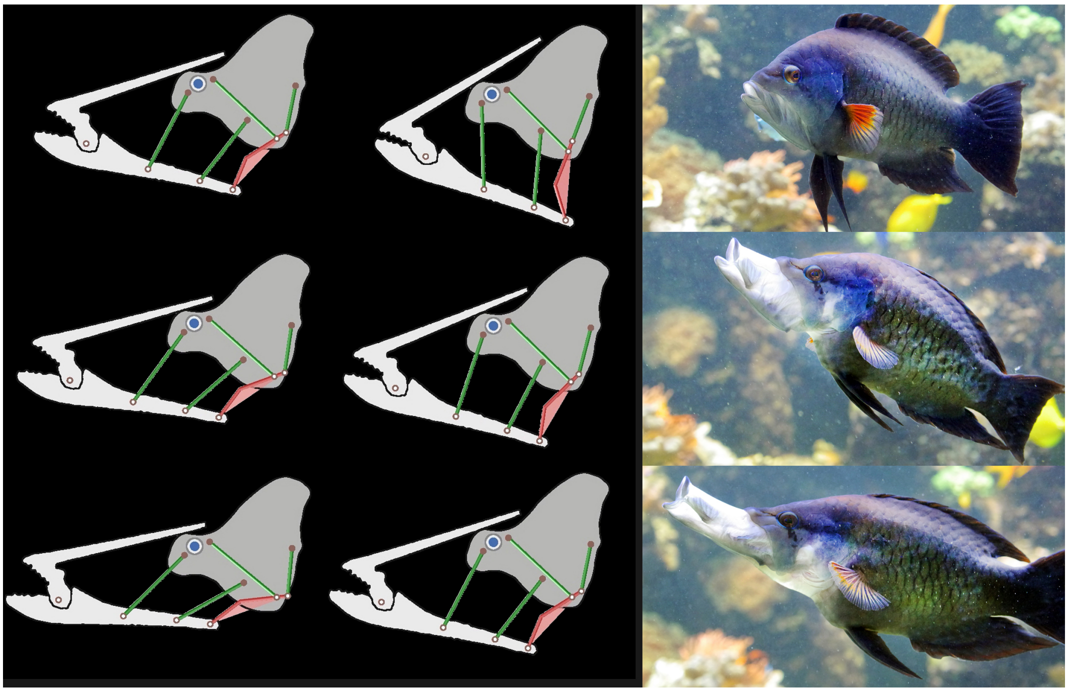

Figure 2.

Figure 3.

On some Curves Related to Pencil of Triangles in Isotropic Plane

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We observe a pencil of triangles in isotropic plane whose vertices A and B are fixed, while vertex C moves on the circumscribed circle. We prove that the locus of all centroids of triangles in such a pencil is a circle, the locus of all Gergonne points is a curve of order 4 and the locus of all symmedian centers is an ellipse. For the tangential triangles of all the triangles in such a pencil we prove that the locus of all centroids of all triangles is a line, the locus of all Gergonne points is an ellipse and the locus of all symmedian centers is a curve of order 3.

Key words: isotropic plane, pencil of triangles, centroid, Gergonne point, symmedian center

MSC 2010: 51N25

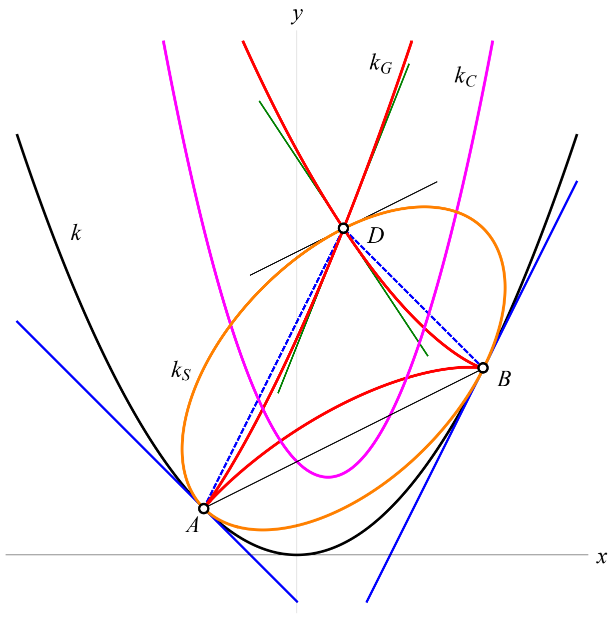

Figure 1: The curve of centroids k_C, curve of Gergonne points k_G and curve of symmedian centers k_S for a pencil of triangles with the same circumscribed circle k

Parallelograms in Vakarelov Quasigroups

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In this presentation we examine an idempotent medial quasigroup which satisfies the identity $ab \cdot b = a$, the so-called Vakarelov quasigroup. We present also some examples of Vakarelov quasigroups. We prove that a Vakarelov quasigroup can be defined in a different way by using a small number of properties.

We introduce the concept of a parallelogram in a Vakarelov quasigroup by means of auxiliary points like in a medial quasigroup. This definition allows direct definition of a parallelogram in a Vakarelov quasigroup.

Key words: Vakarelov quasigroup, parallelogram

MSC 2010: 20N05

Compositional Principles between Architectural Plans and Reality

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The most famous architect of antiquity Vitruvius wrote in his book that quality architecture must meet three key criteria: strength, usability and beauty. The first two are achieved by choosing the appropriate material, the design of the structure, the arrangement of spaces, devices, installations ... Third - the beauty is achieved by choosing the colour, texture, pattern distribution and arrangement of architectural elements. The architectural elements are translated into geometric elements, and according to composition and proportional principles, they are combined into a harmonic whole.

The compositional and proportional principles can be followed since the earliest civilizations. The proportional principles were based on the ratio of small integers, geometrical principles and other important ratios like the irrational number π and the golden section. Especially the golden section is considered as the basic principle of orderliness and beauty, as it can be found both in nature and in the well-established and recognized architecture. Compositional principles are most pronounced in palaces, monuments and sanctuaries, since it has been said from the outset that these buildings must be the most beautiful. Knowledge of proportionality and composition was a carefully guarded secret that the architects retained for themselves.

In today's analysis of the compositional principles of recognized architecture, we try to determine which proportions the architect used to achieve the harmony of the building. We usually encounter several problems here. The first problem is obtaining the appropriate plans of the already built building. Using modern methods of measurement (digital orthophoto, 3D scanning) we can obtain accurate data on the dimensions of the building.

Today it is difficult to get original plans of how did the architect imagine the building. Only the original plans can explain the basic idea of planning. The difference between drafts and plans and subsequent actual implementation may be large. Another problem is related to dimensional tolerance in the analysis of compositional principles. This may be due to the use of standardized building materials, the operation of the material, other influences, etc.

This presentation tries to answer and clarify issues related to the problem of determining the compositional principles of architecture. As an attempt of a compositional analysis of quality architecture, architect Jože Plečnik's church of St. Michael on Barje is presented. This building is on the list to become the part of UNESCO heritage. For the building, original plans are preserved in several versions, then the chronicle of the building, and the measurement of the current state. The results show that the building is an interesting treasure trove of various compositional principles based on proven canon of proportions.

Key words: geometry, composition, proportions, architecture, Plečnik

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Meta-Geometry in Spatial Design
Perceptual Space Analysis in Chinese Landscape Architecture

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This work tries to analyze some selected cases of Chinese Landscape Architecture, intended as space phenomena with reference to the perceptual space related to perspectival perception.

It is generally known that the classical Western graphic analysis and design in architecture was based on geometry knowledge. Most of these activities were also based on some fixed points perspective, as another indicator of the anthropocentrism. It is also generally known that Chinese traditional landscape painting was a typical kind of artistic production of scattered perspective, based on certain guidelines and principles to create architectural landscapes, usually named *Chinese gardens*, and landscape paintings in the ancient China.

This work will propose a retrospective research focusing on the philosophical roots of the mentioned principles. The aim is to reveal the essence of those scattered perspective, either in 2D space (graphic), also considering its interactivity effects, or when the core idea was realized as a perceptual space in real 3D space organizations. Moreover these principles pertain to a wider cognitive level, which won't be limited by classic geometry, but is also related to the non-Euclidean fields, as well as to several non-strictly-geometrical aspects of the space, what we define in terms of *Meta-geometry* here.

Series of graphical analyses of the selected phenomena in Chinese landscape architecture will be presented, aiming to translate this idea into visual diagrams. This will improve the understandability of these spatial systems, moreover, it might provide some inspiration and maybe new ideas for the graphic representation of contemporary architecture and spatial design in this field, as well as a comparison between the Western and the Eastern points of view in relation to landscape architectural design.

The topic presented here is a part of a PhD program (*Architectural, Urban and Interior Design*) carried out at the Department of Architecture and Urban Studies of the Politecnico di Milano.

Key words: perceptual space, graphic analysis, Chinese landscape architecture, perception-phenomenology, visual psychology

MSC 2010: 00A66, 51N05, 01A05, 97U99

An Example of Case Study: Average Visual Facade and Action Guiding
The Main Temple of Qianwei Confucian Temples, in Sichuan, China
Location: South-West area of China
Age: Established in Song Dynasty

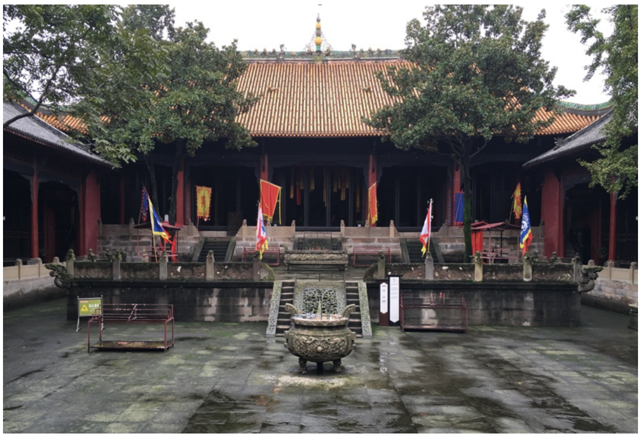

Figure 1: Average Visual Facade (view from courtyard, distance of sight: 15-20m)

Figure 2: Action Guiding (view from Human-action scale, distance of sight: 2-5m)

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Equiareal Orthogonal Patterns on the Sphere

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Orthogonal equiareal patterns on a surface are defined to be coordinate patches such that the first fundamental form is

$$z\,dx^2 + (1/z)\,dy^2.$$

Uniformly distributed coordinate curves split the surface into curvilinear quadrilaterals of one and the same area. These patterns correspond to principal stress lines of plastic deformations under the Tresca yield condition [5, 6].

Based on papers [1, 2, 3], the talk deals with orthogonal equiareal patterns on the unit sphere. They correspond to solutions of the integrable partial differential equation

$$z_{yy} + (1/z)_{xx} + 2 = 0.$$

Symmetry invariant solutions correspond to orthogonal equiareal patterns implicit in the work of R. Lipschitz [4]. The talk focuses on detailed description and explicit construction of the Lipschitz patterns.

Key words: orthogonal equiareal pattern, constant astigmatism equation.

MSC 2010: 35Q53, 35Q74, 53A05

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Algorithmic Classification of Compact Fundamental Polygons for Plane Groups (Poincaré-Delaunay Problem)

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Joint work with Zoran Lučić and Nebojša Vasiljević, University of Belgrade, Faculty of Mathematics

H. Poincaré (1882) attempted to describe a plane crystallographic group in the Bolyai-Lobachevsky hyperbolic plane $\mathbf{H^2}$ by appropriate fundamental polygon. He extended this initiative to space as well. B.N. Delone (Delaunay) and his colleagues in 1960's refreshed this very hard topic for Euclidean space groups, as the stereohedron problem, namely: *To give all fundamental domains for a given space group*; with few results, for particular space groups.

A. M. Macbeath (1967) completed the initiative of H. Poincaré in classifying the 2-orbifolds by giving each with a signature. That is described by a base surface with orientable or non-orientable genus; by some singular points on it, as rotational centers with given periods; by some boundary components, in each with given dihedral corners. All these are characterized up to an equivariant isomorphism, also indicated in this talk.

There is a nice curvature formula that describes whether the above (good) orbifold, i.e. co-compact plane group (with compact fundamental domain) is realizable either in the sphere $\mathbf{S^2}$, or in the Euclidean plane $\mathbf{E^2}$, or in the hyperbolic plane $\mathbf{H^2}$, respectively.

Our initiative in 1990's was to combine the two above descriptions; namely, how to give all the combinatorially different fundamental domains for any above plane group. Z. Lučić and E. Molnár completed this by a graph theoretical tree enumeration algorithm. That time N. Vasiljević implemented this algorithm to computer (program COMCLASS), of super-exponential complexity, by certain new ideas as well. This cooperation started in time of Yugoslav Geometrical Seminars (1983, Arandjelovac, as I remember on my first participation).

In the sad time of the Yugoslav war we lost our manuscript, then the new one has been surprisingly rejected (?!). Now we have refreshed our manuscript to submit again, and that is to appear as [12] with many unexpected actualities. Here we intend to present a report on it, also with some open problems.

Key words: plane discontinuous group, fundamental polygon, Macbeath signature, Poincaré-Delone (Delaunay) problem

MSC 2010: 57S30, 52C20, 51M10, 68R99

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Isoptic Surfaces of Large Concave Polyhedral Meshes

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The isoptic surface of a three-dimensional shape is recently defined by Csima and Szirmai [1] as the generalization of the well-known notion of isoptics of curves, that is collection of points from where the shape can be seen under a predefined angle. In that paper isoptic surfaces of some regular convex polyhedra are also presented. However, the computation of isoptic surfaces by that algorithm requires extensive computational time and CAS resources (*Wolfram Mathematica*). Moreover, the method cannot be extended to concave shapes. In this talk, based on [2], we present a new algorithm to find points of the isoptic surface of a triangulated shape, which works for large convex and concave polyhedral meshes as well. Alternative definition of the isoptic surface of a shape is also presented, and isoptic surfaces are computed based on this new approach as well.

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Domino Tilings of Euclidean and Hyperbolic Boards

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Several studies deal with tilings with squares and dominoes of the well-known regular square mosaic in Euclidean plane, but not any with the hyperbolic regular square mosaics. In the presentation, we examine the tiling problem with colored squares and dominoes of one type of the possible hyperbolic generalization of $(2 \times n)$-board. The recurrence sequence describing the number of the different tilings on this board is a fourth order linear homogeneous recurrence sequence and it is a generalization of the Fibonacci sequence, which has also connection to the tilings.

In the presentation, we also introduce the 3-dimensional Euclidean $(2 \times 2 \times n)$-board on which we examine the tiling problem with colored cubes and bricks.

Key words: domino tiling, hyperbolic mosaic, tiling in space, generalized Fibonacci sequence

MSC 2010: 52C20, 05B45, 05A19

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A Convergent Triangle Tunnel

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In triangle geometry, only a few *convergent* infinite sequences of triangles associated to a given triangle $\Delta_0 = A_0B_0C_0$ are known. In the beginning, we construct a closed path of which there exists at least one in each triangle: Starting at an arbitrary point P_0 on an arbitrary side, say A_0B_0, we draw the orthogonal line until we hit the adjacent side where we proceed in the same way. The procedure ends if we arrive at $P_3 \in A_0B_0$. Usually, P_0 and P_3 will be two different points and the mapping $\pi : P_0 \mapsto P_3$ is projective for it is the composition of three perspectivities since the three perspectors are collinear and the ideal point of $[A_0, B_0]$ is self-assigned, there exists only one fixed point A_1 of π that gives rise to a closed triangular path $\Delta_1 = A_1B_1C_1$ inscribed into Δ_0, unless Δ_0 is isosceles.

The vertices of Δ_1 arise independently of the choice of the side on which we start. It turns out that Δ_0 and Δ_1 are similar. Clearly, further nested triangles Δ_i (with $i \in \{2, \ldots, n\}$ similar to Δ_0 show up. The factor $\lambda < 1$ of similarity between any pair of subsequent triangles is constant and a cyclic symmetric function of Δ_0's side lengths. Thus, the path $A_0A_1A_2A_3\ldots$ consists of subsequent orthogonal edges whose lengths form a geometric sequence. This allows us to find the limit $L := \lim_{n\to\infty} A_n$ (which is at the same time the limit position of the other vertices). It turns out that L is an yet unknown rational triangle center that is also the radical center of the three Thales circles on the segments A_0A_1, B_0B_1, and C_0C_1.

Key words: triangle, closed path, geometric sequence, triangle center

MSC 2010: 51Nxx, 51Fxx

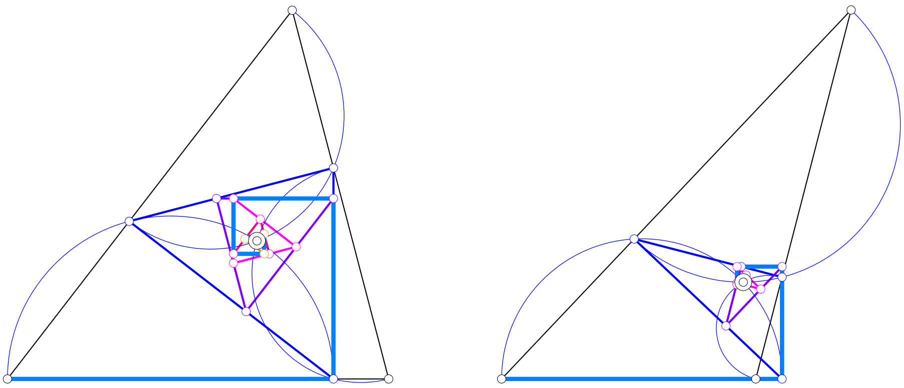

Figure 1: The polygon $A_0A_1A_2\ldots$ is a discrete logarithmic spiral for acute and obtuse triangles.

Ω-surfaces and Weierstrass Type Representations

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The Weierstrass-Enneper representation [10] has been used extensively in surface theory to create interesting examples of minimal surfaces in $\mathbb{R}^3$. More recently Weierstrass type representations have been developed for creating surfaces of constant mean curvature in other 3-dimensional space forms. For example, the Bryant representation [3] for CMC-1 surfaces in $\mathbb{H}^3$. In [2], a Weierstrass type representation was developed for certain marginally trapped surfaces in Minkowski space $\mathbb{R}^{3,1}$, which unifies most of these known Weierstrass type representations.

In 1911, Demoulin developed a class of surfaces called Ω-surfaces. These are surfaces that envelope isothermic sphere congruences. Examples of these include isothermic surfaces and linear Weingarten surfaces. One characterisation of Ω-surfaces is the existence of a dual surface (see [6]), induced by the Christoffel transformation of its isothermic sphere congruences.

L-isothermic surfaces, that is surfaces which admit curvature line coordinates that are conformal for the third fundamental form, are Ω-surfaces and have a special characterisation in terms of dual surfaces. On the other hand, surfaces that admit Weierstrass type representations are L-isothermic surfaces, see [4, 5]. In this talk we shall explore this relationship and give a new interpretation of Weierstrass type representations in terms of dual surfaces.

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Harmonic Evolutes of B-scrolls with Constant Mean Curvature in Minkowski Space

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A ruled surface in 3-dimensional Minkowski space $\mathbb{R}^3_1$ is a surface which can be parametrized by

$$\mathbf{f}(u,v) = c(u) + ve(u),$$

where c(u) is a base curve, $c\colon I \to \mathbb{R}^3_1$, $I \subseteq \mathbb{R}$ open, and $e(u)$ a non-vanishing vector field along c which generates the rulings $v \in \mathbb{R}$. When $c'(u)$, $e(u)$ are both null, ruled surfaces are called the null-scrolls, or in the special case, the B-scrolls.

In this presentation we investigate properties of harmonic evolutes of B-scrolls with constant mean curvature in Minkowski space and their relationship to null Bertrand curves. The harmonic evolute of a surface is the locus of points which are harmonic conjugates of a point of a surface with respect to its centers of curvature p_1 and p_2. The Bertrand curves are curves whose principal normals are the principal normals of another curve.

Basic properties of harmonic evolutes of surfaces in Euclidean space have been investigated in [1], of surfaces in Minkowski space in [2] and of null Bertrand curves in [3, 4].

Key words: Minkowski space, harmonic evolute, B-scroll, Bertrand curve

MSC 2010: 53A35, 53B30

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Geometry – Design – Textile – Fashion

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As a lecturer of mathematical courses (mathematics and constructive geometry) at the Faculty of textile technology, I tried additionally to explore the connections and the possibilities of combining mathematics (especially geometry) with textile profession, especially through the preparation of the topics for final and graduate thesis. In this presentation, I would like to present some directions of that research, as well as some of the results.

Ratios, scales and proportions, symmetry and asymmetry in designing of garments, fashion accessories or textile patterns; Geometry of lace and embroidery patterns on the folk costumes; Geometric figures as elements of the garment construction; Geometric shapes and golden section and their use in textile design, design of garments, fashion collections or fashion accessories ... these are just some of the topics that will be discussed.

Key words: mathematics, geometry, design, textile, fashion

Regularizing Quadrangles in the Möbius Plane

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The presentation is based on a joint work with G. Eberharter and J. Lang. For any given cross ratio $\delta \in \mathbb{C}$ we define a non-linear Möbius-invariant procedure creating vertices of a new quadrangle (successor) from a given quadrangle. With respect to Möbius transformations a quadrangle bears just one invariant - its cross ratio $z \in \mathbb{C}^* := \mathbb{C} \cup \{\infty\}$. We regard this cross ratio of a quadrangle and of its successor. Reiterating this process yields a series of cross ratios $(z_i)_{i\in\mathbb{N}}$ for the different generations. Surprisingly, we discover a regularizing property: This series, for generic input data, will tend to some limit cross ratio belonging to Möbius-regular quadrangles. These special cross ratios are either ∞ or $1/2$. Figure 1 displays an example for the latter case with limit value $\lim_{i\longrightarrow\infty} z_i = 1/2$.

As the values of the series $(z_i)_{i\in\mathbb{N}}$ are generated iteratively by a rational cubic function $R(z)$ in one complex variable our problem leads to the theory of dynamics in one complex variable and the existence of attracting fixed points. The values $\delta \in \mathbb{C}$ for which this phenomenon of regularization develops are examined; they belong to well-defined disjoint open subsets $C_{0,1} \subset \mathbb{C}$.

The regularizing effect also depends on the shape of the starting quadrangle. This can be characterized with the help of complex dynamics:

Theorem 1 *The procedure defined above, employed iteratively, has the following regularization properties:*

- *If the construction cross ratio δ belongs to the open subset $C_0 \subset \mathbb{C}$ and the shape z_0 of the initial quadrangle provides a value within in the Fatou set $\mathcal{F}$ of R, the procedure regularizes towards the shape $z^* = \infty$. This is the shape of a quadrangle with $\mathbf{p}_0 = \mathbf{p}_2$ and $\mathbf{p}_1 = \mathbf{p}_3$.*
- *If the construction cross ratio δ belongs to the open subset $C_1 \subset \mathbb{C}$ and the shape z_0 of the initial quadrangle provides a value within in the Fatou set $\mathcal{F}$ of R, the procedure regularizes towards the shape $z^* = 1/2$. This is the standard case of regularization as illustrated in Figure 1.*

In both cases the exceptional shapes z_0 (where no regularization happens) are exactly those lying in the corresponding Julia set $\mathcal{J}$ of R (see a typical example in Figure 2). For the respective values δ any quadrangle with generic shape z_0 is regularized by our procedure.

Key words: Quadrangles in the Möbius plane, regularizing procedures, regular quadrangles, discrete dynamics

MSC 2010: 51N30, 37F10, 37F40

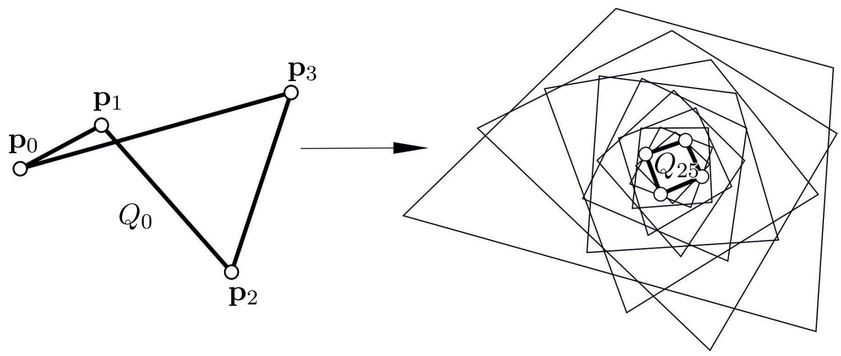

Figure 1: The initial quadrangle $Q_0 = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ and some of the next 25 generations (enlarged and shifted) for $\delta = 2.5 + i$. Here we get $z_0 \approx 1.168 - 0.345i, \ldots, z_{24} \approx 0.472 + 0.013i, z_{25} \approx 0.526 - 0.007i$.

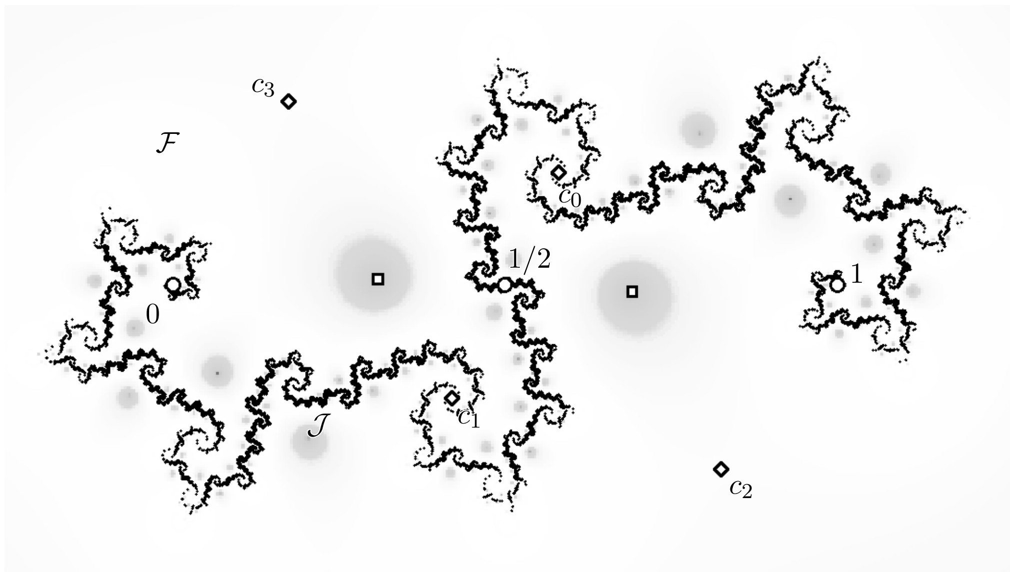

Figure 2: The Julia set $\mathcal{J}$ for $\delta = 0.5 + 0.49\,(\cos 3\pi/8 + i \sin 3\pi/8) \in C_0$. Here $t_0 = \infty$ is the attracting fixed point, the three marked points 0, 1/2 and 1 belong to $\mathcal{J}$. The diamond-shaped symbols indicate the critical points $c_0, \ldots, c_3$, the squares mark the two finite preimages of ∞.

Logarithmic Spirals in Kinematics

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The logarithmic spiral is an isogonal trajectory of a pencil of lines. This gives rise to a series of geometric consequences, and we pick out a few which are relevant for kinematics: When a logarithmic spiral rolls on a line, its asymptotic point traces a straight line, too. When involute spur gears are to be generated by virtue of the principle of Camus, the auxiliary curves must be logarithmic spirals. Two congruent logarithmic spirals can roll on each other while their asymptotic points remain fixed (syntrepent curves). A composition of two such rollings gives a two-parametric motion which allows a second decomposition of this kind.

Some of these properties hold similarly in spherical kinematics. For example, spherical loxodromes are auxiliary curves for involute bevel gearing. Dualization, according to Study's principle of transference, leads even to ruled surfaces as tooth profiles for gears with skew axes.

Key words: logarithmic spiral, involute spur gears, spherical loxodrome, involute bevel gears

MSC 2010: 53A17

Figure 1: Stairs climbing wheels (collection of mechanism models, TU Vienna)

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Monostable and Bistable Auxetics Structures

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In this paper, we will analyze on a macroscale monostable auxetic structures whose movements can be described geometrically as well as bistable ones. We investigate and elaborate auxetic behavior in a purely geometric way that is based on the kinematic movement of different frameworks. We demonstrate its usefulness by analyzing the involved geometry with computer software but without computer simulations or numerical approximations. Using cut flat material and, depending on the cuts and the material used, we will enable a kinematical movement of the structures. Bistable structures owe their mobility to the elasticity of the material. Based on geometric considerations, we combine rigid materials and composites and appropriate joint connections will allow the application of this system in an architectural scale.

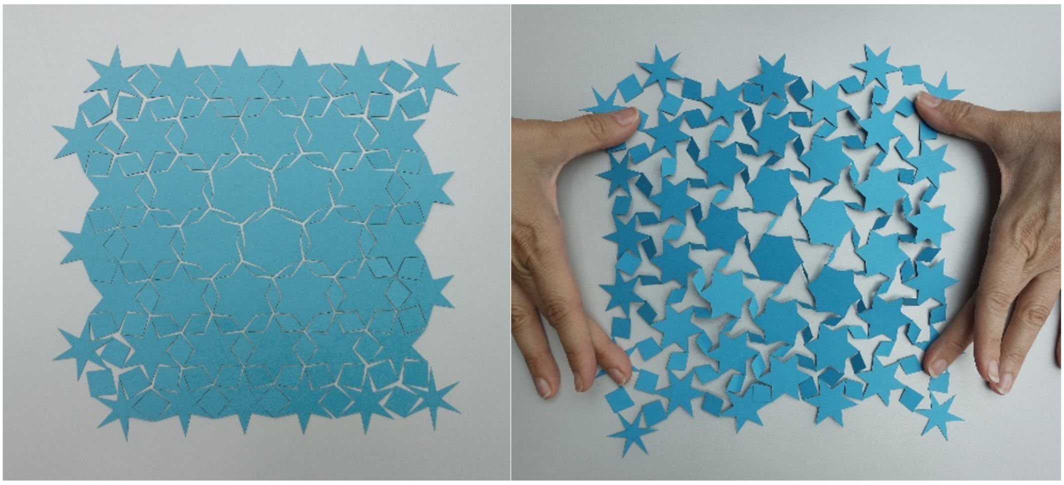

Figure 1: Bistable auxetic structure

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Workshop "Parametric Modelling and Digital Fabrication"

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Parametric modeling and digital fabrication are very relevant topic in the training of students for future careers in the architecture, engineering and construction industry. Although various novel education programs specialize in digital fabrication, relatively little has been written on concepts for a deeper integration of digital technologies in the engineering curriculum. This paper describes the teaching approach and results of the workshop "Parametric Modelling and Digital Fabrication" held in Zagreb. We will draw out geometric aspect and didactical approach that lead us to motivate students and to finish workshop with awesome results.

The workshop "Parametric Modelling and Digital Fabrication" was held at the Department of Civil Engineering at the University of Applied Sciences in Zagreb from February 26th to March 2nd, 2018. The aim of this five-day workshop was to introduce students to parametric modelling and to CNC fabrication process using a laser cutter. The international workshop was intended for students of civil engineering at Zagreb University of Applied Sciences and for guest students of architecture from the Faculty of Technical Sciences, University of Novi Sad, Faculty of Architecture, Civil Engineering and Geodesy, University of Banja Luka and University Nikola Tesla, Belgrade.

The topic of the workshop was pavilion, with the waffle slab constructive system, which allowed both students of civil engineering and architecture to find their own motivation to build their own pavilion beginning with a virtual parametric design and ending with an analogue scale model.

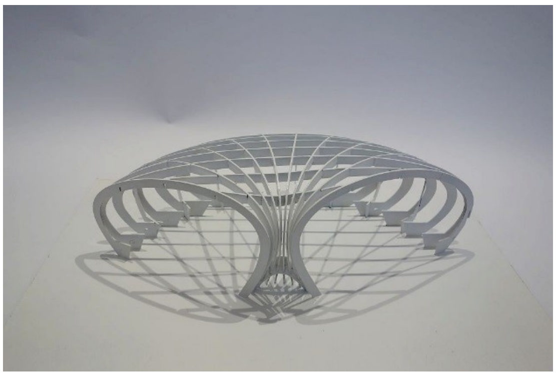

Figure 1: One of the results of the workshop

MSC 2010: 97G40, 97G80, 97Q60

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Descriptive Geometry Education in Architecture with Mobile Augmented Reality Application

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Augmented reality technology(AR) is no longer a mysterious new technology. Given the Google and Apple and other technology companies to invest in AR technology, people can now easily use smart devices (mostly mobile phones) to try the AR technology brought about by the new experience. People can get a 3D- additional-layer based on real objects through mobile phones. In this research, we describle the implementation and evaluation of an application of augmented reality in the visualization of 3D models and the education of descriptive geometry in architecture for students who are more familiar and interested in new technologies. Descriptive geometry allows representation of three-dimensional objects in two dimensions, by using a specific set of procedures. One scope of traditional Descriptive Geometry was and still is, indeed, supporting the understanding of the spatial properties and relations, which for a student in architecture is one of the most relevant educational goals. Another scope was making students able to figure out and control the architectural space starting from planar images set as graphic projections of that space. However, sometimes the link between three-dimensional space and bidimensional graphics remained obscure to the learners, with negative consequences on the development of their spatial imagination. Computer graphics, augmented reality and virtual reality show very clearly, nowadays, this link. Last but not least, the attractiveness of the spatial simulations is also a point to take into account in the prospective educational program in our field.

In the oral presentation we will focus on some pilot projects and tests, as reference examples for reconsidering some aspects of graphics education in the field of architectural design at various scales. In this sense, the educational context of the PhD program of "Architectural, Urban, and Interior Design" at the Department of Architecture and Urban Studies of the Politecnico di Milano, and the PhD educational level in general, is an appropriate wheel between basic and advanced education, as well as between basic and advanced scientific research and investigation and professional world. That is also the reason of this co-authorship.

Key words: AR, architecture visualization, descriptive geometry, geometry education

MSC 2010: 00A66, 51N05, 01A05, 97U99

Figure 1: Demo of application AR in architecture representation

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Caustics of Free-formed Curves

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Let's consider a curve as a mirror. If we emit parallel light rays onto the curve, than the reflected rays generally have an envelope. This envelope is called the caustic of the mirror curve. In this lecture the caustic curve of a free-formed curve and its alteration by the modification of a control point are studied. We will provide all of the results in a general parametric form. Some case studies in terms of Bézier and B-spline curves are also discussed.

Key words: Bézier curve, B-Spline curve, free-formed curve, caustic curve, envelope

MSC 2010: 68U05

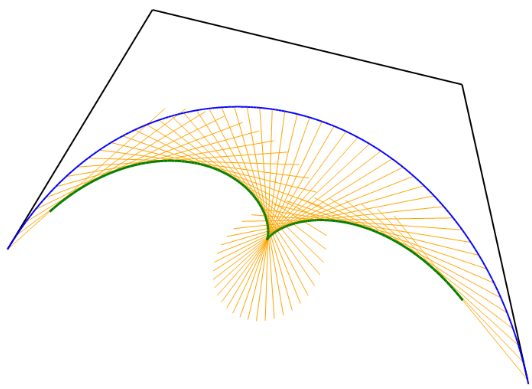

Figure 1.

Measures

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Here are few reflections on the attempts to confront mathematical and artistic approach to measuring various aspects of creativity, imagination and our understanding of aesthetic values hidden in the produced artifacts in fine art and in science, mathematics in particular. Phenomena that are carriers of these subtle values enable comparison of the concepts of measures and measurement strategies in maths and art. Short revision of the surface measure definitions and its application within the two environments is presented, together with continuous efforts of finding their common features and differences.

Measure is a certain limit that can be reached in-between two different measurements based on similar, anyhow not completely equal principles. Limit, in which inner and outer measures meet and overlap finding thus an agreement reflected in the real value, does not exist necessarily in all cases.

Measure of an empty space is zero. Just the measurement units, objects filling the space, determine the non-zero measure value. Consequently, these objects themselves, items representing our expectations, define the final measure value. The preconditions defined as assumptions influence a sort of the measure quality or type. Mathematics states these clearly by determining initial conditions related to the specific desired type of measure, which are generally agreed and accepted and they serve as certain guide for related measurements. Several measures can be defined, such as Jordan, Borel, arithmetic, Dirac, Carathéodory or Lebesgue-Stieltjes measure, and we can speak about regular, complete, finite, σ-finite, arithmetic, and many other measures, which differ in the measuring approach and the measure quality for which they are applied.

Do similar laws exist also in art, or is the imagination space of an artist completely unlimited? Are there certain obstacles, defined by authors themselves, or imposed intuitively by the level of mankind culture development that are influencing creative work on the masterpiece, in the sense of its pre-defined inner desirable measure? Are the different measures reflected in different artistic styles approaching different aims and looking for different qualities?

It is a pursuit of elegance that captures essence, and gives us a precise insight on relations.

Xah Lee

Key words: mathematics and art, interior and exterior measure, truth, beauty

MSC 2010: 28A99, 91F99, 97M80

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A Set of Planar and Spatial Tessellations Based on a Compound 3D Model of the 8D Cube

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The 3-dimensional framework (3-model) of any k-dimensional cube (k-cube) can be produced based on starting k edges arranged by rotational symmetry, whose Minkowski sum can be called zonotope. Combining $2 < j < k$ edges, 3-models of j-cubes can be built, as parts of a k-cube. The suitable combinations of these zonotope models can result in 3-dimensional space-filling mosaics. The investigated periodical tessellations always hold the 3-model of the k-cube and necessary j-cubes derived from it.

The mosaics can have fractal or fractal-like structure as well, since the stones can be replaced with restructured ones. The hulls of 3-models of k- and $3 < j$-cubes can be filled with different sets of 3-models of $2 < j < (k-1)$ or $2 < i < (j-1)$ -cubes touching each other at congruent faces. Another possibility is if the 3-models of the given k-cubes and of the derived j-cubes are arranged along the outer edges of the restructured models and the faces are replaced with central symmetrically arranged sets of the above elements. The inner space of the new compound models is filled also with the initial models. If also the inner edges are followed by 3-models of the k- and j-cubes, the construction can require unmanageable amount of the elements, from practical points of view, but the restructured mosaics can have a more consequent fractal like structure.

The intersections of the mosaics with planes allow unlimited possibilities to produce periodical symmetric plane-tiling. Moving intersection planes results in series of tessellations or grid-patterns transforming into each other. These can be shown in varied animations.

Key words: constructive geometry, hypercube modelling, tessellation, fractal

MSC 2010: 52B10, 52B12, 52B15, 65D17

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The Three-Reflections-Theorem Revisited

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It is well-known that, in a Euclidean plane, the product of three reflections is again a reflection, iff their axes pass through a common point. For this "Three-reflections-Theorem" (TRT) also non-Euclidean versions exist. The lecture presents an affine version of it, considering a triplet of skew reflections with axes through a common point. It turns out that the essence of all those cases of TRT is that the three pairs (axis, reflection direction) of the given (skew) reflections suit into an involutoric projectivity. For the Euclidean case and its non-Euclidean counterparts this property is automatically fulfilled.

From the projective geometric point of view a (skew) reflection is nothing but a harmonic homology. But while in the affine situation a reflection is an indirect involutoric transformation, "direct" or "indirect" makes no sense in projective planes. A harmonic homology allows an interpretation as both, as an axial reflection and as a point reflection. Nevertheless, one might study products of three harmonic homologies, which result in a harmonic homology again. Some special mutual positions of axes and centres of the given homologies lead to elations or even to the identity, too.

A consequence of the presented results are further generalisations of the TRT, e.g. in an affine 3-space or in circle geometries.

Posters

Grapho-Analytical Framework for Geometric Supersymmetry

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Nowadays, the rapid development of computer technology, together with the accompanying software packages, enables more efficient analysis of geometric forms, both planar and spatial. The research of geometric transformations represents an important task, since they provide numerous possibilities for the generation of new geometric entities. This research introduces a particular geometric transformation, herewith called Supersymmetry, which is the composition of two constitutive elementary transformations: a perspective collineation and an inversion (harmonic symmetry) simultaneously applied to the domain and codomain of the perspective collineation. The perspective collineation in space is a central projection from one plane into another [1, 2, 5], while the inversion is a mapping based upon the polarity of points with respect to a circle and its centre [2, 3, 4]. Thus, the supersymmetry becomes a mapping with two fundamental elements: the absolute, being an arbitrary sphere (as a surface of transformation), and the pole, being an arbitrary point incident to the absolute. In a particular case, when a planar curve, that is to be transformed, is coplanar with the pole, the absolute becomes a circle (which is the intersection of that plane and the sphere). The arbitrary point A is mapped through the following procedure: (i) the determination of the intersecting point P between the absolute and the transformation ray, defined by the pole and the point A; (ii) the determination of the point A', being centrally symmetrical to the point A with respect to the point P, which is the corresponding point to the initial point A. The method is valid for any set of points representing either plane curve, space curve or curvilinear grid on a surface.

In this work, it has been proved that the major property of the supersymmetry, that is the relation between corresponding points (a pure Euclidian central symmetry), is independent on the choice of the constitutive perspective collineation. On the basis of the underlying grapho-analytical reasoning, the mapping which describes supersymmetry has been analytically treated. Hence, the procedure for the computer aided application has been created, and it is applicable both to curves and surfaces. Such a procedure provides most complete insight into geometric structure of the obtained geometric form and thus the possibility of choice of the surface's curvilinear grid that is to be mapped. It also enables an interactive investigation along the process of visual presentation, and therefore an easier prediction of transformed shapes.

Furthermore, the procedure has been applied to the mapping of ellipse and ellipsoid, being examples of planar and spatial transformation, respectively. It has been shown that supersymetry, regarding the mutual position between ellipses and the pole, results in the family of oval-like curves only when the pole coincides with the axis of an ellipse (Fig. 1). Otherwise, the obtained curves are of an asymmetric shape. Analogously, ellipsoids are being transformed in either symmetric or asymmetric ovoid-like surfaces (Fig. 2).

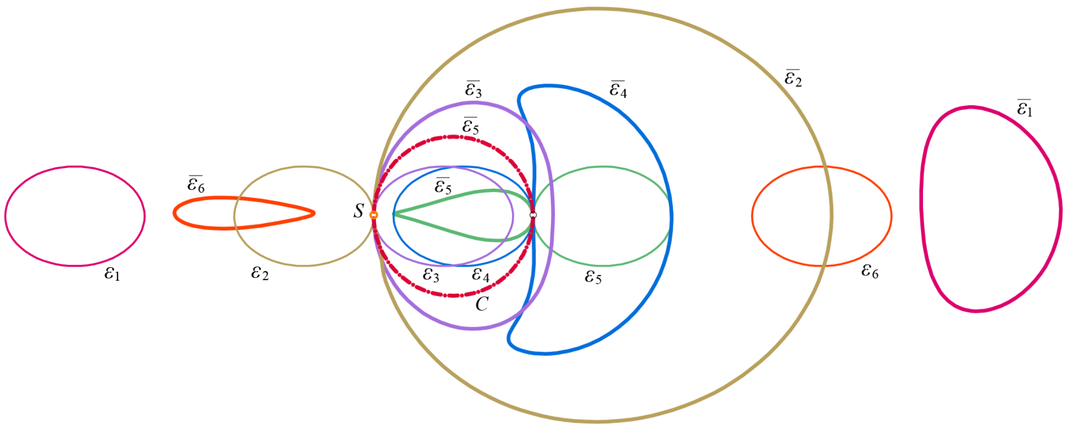

Figure 1: The supersymmetry of ellipse in various positions regarding the pole S and the absolute C

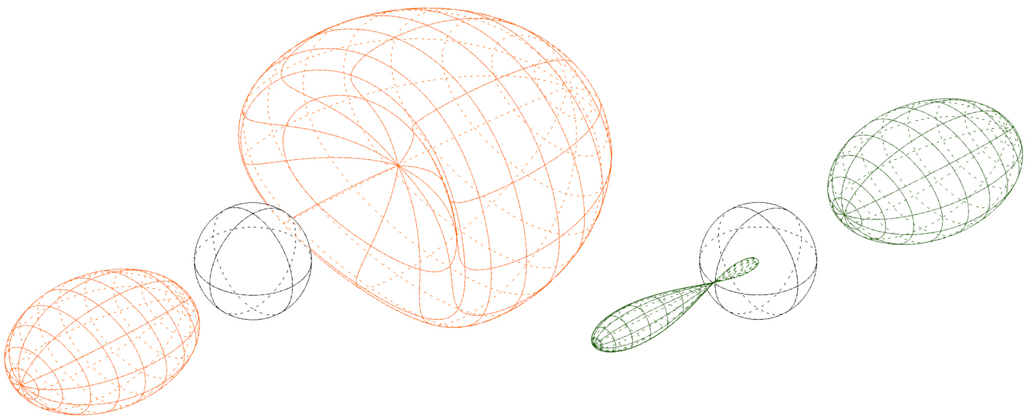

Figure 2: The supersymmetry of three-axial ellipsoid in two positions regarding the pole and the absolute

Key words: supersymmetry, inversion, perspective collineation, geometric transformation, mapping, analytical modelling

MSC 2010: 14N05, 14Q05, 14Q10, 68U05

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Shades and Shadows for Architecture and Design Students

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The scope of descriptive geometry courses was radically cut down as the Bologne process was introduced at the Faculty of Architecture, University of Zagreb. Most students come across descriptive geometry courses for the first time at the first year of studying. Because of a limited time within the teaching program, we aim to obtain the application of theoretic geometric problems in the shortest time possible. Through this studying process students find the area "Shades and Shadows" the most difficult to overcome. In order to facilitate the understanding and the application of this area, we present the acquired knowledge through several educational exercises.

We dedicate this work to our dear colleague and teacher Nikoleta Sudeta (1949-2018).

Key words: descriptive geometry, shades and shadows

MSC 2010: 51N05, 97G80

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