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ABSTRACTS

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Plenary lectures

Advancing mathematics education: Integrating learning analytics and AI for effective learning design and assessment

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In the realm of mathematics education, fostering responsible, creative, and curious students is paramount. This presentation explores the integration of learning analytics, meaningful learning design, innovative pedagogical approaches, and the collaboration between human educators and artificial intelligence (AI) to enhance mathematics education. Through a review of the author's key recent studies, this presentation aims to contribute insights into the advancements in this field.

Teachers play a pivotal role in shaping students' learning experiences, and their decisions in learning design significantly impact students' learning outcomes, [3]. A comprehensive concept and tool called Balanced Learning Design Planning (BDP) has been proposed, emphasizing learning outcomes, constructive alignment, assessment validity, and the integration of learning analytics, [6]. The presentation also highlights the availability of the free-to-use BDP collaborative software tool at learning-design.eu.

Learning analytics dashboards serve as essential tools for providing students with insights into their learning progress and facilitating reflection and adaptation of learning plans and habits, [2]. Students highly value features that aid short-term planning and organization of learning, while cautioning against comparison and competition, which can be demotivating, [2]. By integrating AI-powered analytics, educators can offer students valuable insights into their learning progress, supporting personalized learning experiences while being aware of potential downsides.

Assessment plays a critical role in guiding learning, and its integration within learning design requires careful consideration to ensure its validity, [1], reliability, fairness, and acceptability. Student perspectives on e-assessment in mathematics underscore the importance of student-centered approaches and pedagogical alignment, [4]. To further enhance assessment practices, the proposal involves allowing students to openly partner with AI chatbots for formative and summative assessments in mathematics, where they solve problems and critically evaluate their interactions with the chatbot.

The COVID-19 pandemic has emphasized the significance of innovative pedagogical approaches such as Inquiry-Based Learning and Flipped Classrooms (FC), [5]. FC, facilitated by AI and technology, can actively engage students and foster collaborative learning. Further research is needed to explore the effective use of FC in online and blended learning environments, specifically in the context of teaching and learning mathematics.

Collaborative efforts among educators across borders and institutions are vital for professional development in mathematics education, [7]. Online platforms (e.g., learn.foi.hr) and AI-powered tools provide avenues for knowledge sharing and the dissemination of best practices, fostering continuous improvement in mathematics education.

In conclusion, the integration of learning analytics, AI, and innovative pedagogical approaches enhances mathematics education by tailoring learning experiences, supporting personalized assessment, and promoting collaborative learning environments. The collaboration between human educators and AI technologies holds promise for advancing mathematics education and equipping students with the necessary skills for the future.

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Integer sequences connecting to some geometric constructions

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In geometry, there are many exercises where we have to solve discrete geometric, combinatorial, or number theory problems. The result is sequence, or, in special case, positive integer sequence. We shall give some sequences, mostly with recurrence relations connecting to some geometric constructions. Three of them are introduced shortly in the following:

Ellipse chains connecting to hyperbola. Let us consider the hyperbola $\mathcal{H}$. We examine special chains of circles and ellipses between the branches of hyperbola $\mathcal{H}$ (or outside $\mathcal{H}$), such that the circles (ellipses) are tangent to the hyperbola $\mathcal{H}$ and mutually tangent to each other (see Figure 1). Furthermore, we give the recurrence relations for the tangential points. We define a tangential chain of ellipses between the branches of $\mathcal{H}$ where the centers of the ellipses coincide with the centers of the circles. We give recurrence relations for the ellipses' parameters. We also define and examine a special chain of ellipses inside the branch $x > 0$ of the hyperbola $\mathcal{H}$ when the ratio of the minor and major axes is fixed.

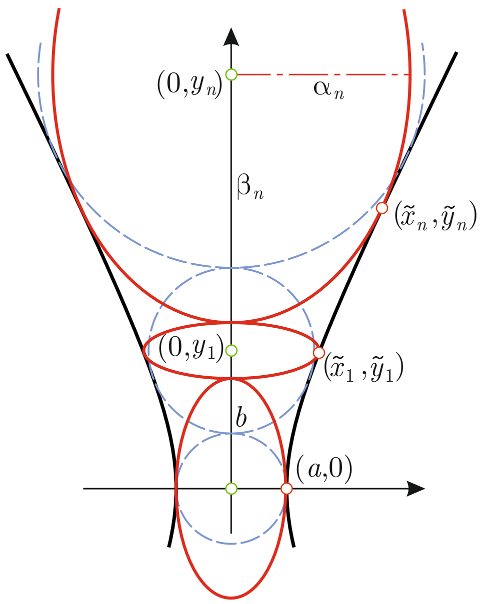

Figure 1: Ellipse and circle chains between the branches of hyperbola.

Sequences associated with a special cube chain. We define a chain of cubes as an infinite part of the cube grid in 3-dimensional space, as in 2. Now we associate the vertices with positive integers, which give the numbers of the shortest paths to the vertex from the base vertex of the first cube. Finally, we shall obtain the sequences (a_i), (b_i), (c_i), and (d_i) with recurrence relations associated with the vertices of the cube chain.

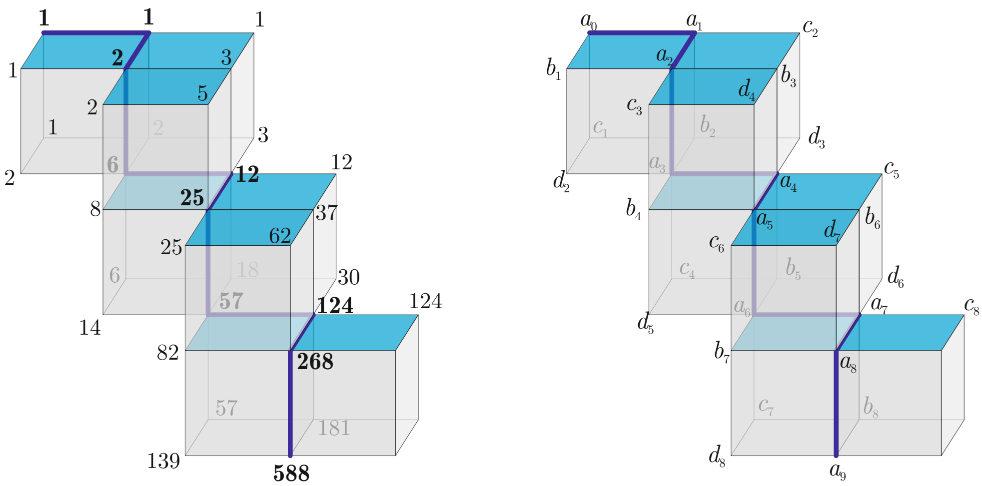

Figure 2: Chain of cubes in a zig-zag form.

Sequences associated with hyperbolic regular mosaics. We introduce the hyperbolic Pascal triangle $\mathcal{HPT}$ and some of its interesting and geometric properties. These arithmetical triangles are based on the hyperbolic regular square mosaics, and some recurrence sequences associated with $\mathcal{HPT}$ are connected to the growing ratios of hyperbolic regular square mosaics.

Key words: recurrence sequence in geometry, integer sequence, ellipse chain, cube chain, hyperbolic mosaic.

MSC 2010: 52C26, 11B37, 97K30, 05A10

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Conformal kinematics at infinity

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Euclidean geometry and non-Euclidean geometries in Cayley-Klein sense can be distinguished by their sets of "ideal points" and, if required, further geometric structures in these ideal points. For example, the ideal points of Euclidean or pseudo-Euclidean geometry form a hyperplane. The additional geometric structure is a regular quadric that has no real points in the Euclidean case. The study of non-Euclidean geometries under consideration of ideal points has been extremely fruitful in the past and brought about numerous important insight, also for Euclidean geometry itself.

Similar ideas have developed over time for concepts of kinematics (and also other mathematical disciplines). The basic idea is to view a smooth set of transformations (a motion, a Lie group, etc.) as point set in some kinematic image space and then study a "suitable closure". The thus added new boundary points encode a surprising amount of information on the original set of transformations. Their relevance is certainly comparable to that of ideal points of curves and surfaces in Euclidean and non-Euclidean geometries.

The talk will feature some recent success stories for this concept of "kinematics at infinity" and then focus on aspects of the group generated by reflections in spheres and planes (conformal transformations). It contains important subgroups like the group SE(3) of rigid body transformations or the group of planar elliptic or hyperbolic transformations. In general, it has a well-structured and regular geometry and it allows for illustration of real phenomena. We will in particular study the displacements at infinity ("null displacements") and their degenerate kinematics which, nonetheless, is meaningful. As one example, we use it to derive a geometric algorithm for the factorization of motion polynomials.

Much of this talk is based on joint work with Zijia Li (Chinese Academy of Sciences) and Johannes Siegele (University of Innsbruck).

Key words: conformal geometric algebra, kinematics, null quadric, rational motion, factorization

MSC 2010: 15A66, 70B10

Contributed talks

Curve of centers of the special conic section pencil

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In the papers [2], [3] using the facts from the conics theory in the Euclidean, pseudo-Euclidean and quasi-hyperbolic planes, we have studied and proved numerous facts related to Steiner's deltoid in various ways. In this paper, the intention is to see what can be said about the curves of the center in some special pencil of conics and to show how, studying the center curves in the special pencil of conics a connection with Steiner's deltoid curve has been found.

Key words: pencil of conics, deltoid curve, curve of centers

MSC 2010: 51M15, 51A45, 51M99, 51N25

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Back to *Incunabula.* Reconsidering orthographic projections in *De Prospective Pingendi*

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Figure 13 in Book I of *De Prospectiva Pingendi* [2], is the indisputable must-reference diagram for the understanding of the projective bases of perspective representation according to Piero Della Francesca's methods.

Aim of the diagram is to explain how the perspective of the square $BDEC$ (which may be considered as a floor plan) is realized, and how other perspective points and lines can be determined to complete the perspective outline of the whole cubic space.

A side view of the cube, including on the left picture plane, the viewing point A, and the related viewing lines, is also integrated in the figure, to support the perspective demonstrations meticuolously carried out in the written text.

Our attention was however attracted by the other point A, namely, that shown inside the big square on the right, as the apparent meeting point of the lines BD and CE, and often considered as the vanishing point of these two lines.

But, at the time of Piero Della Francesca, the vanishing point was not recognized in the form we know it nowadays, or, as the perspective image of a point at infinity of real lines, since the point at infinity itself was not known yet.

What does that point A represent then? And what real lines the segments DA and EA correspond to? In order to find a reasonable answer to these questions, orthographic projections were called into question.

Indeed, although the official and systematic theory was provided by Gaspard Monge about three centuries later, we know that orthogonal views were already used in practice since acient times.

Following this idea, we discovered that orthogonal projections were profoundly embedded in the body of the diagram presented, playing a relevant role in the interpretation of it.

This research work has in part been presented at the online conference Nexus 2021, organized by the TU Kaiserslautern (Germany) [6], and the conference paper has been selected for an extended version published on the *Nexus Network Journal* in 2022 [1].

Key words: Piero della Francesca, De Prospectiva Pingendi, renaissance perspective, perspective, descriptive geometry

MSC 2010: 00A05, 00A66, 51N05, 01A05, 97U99

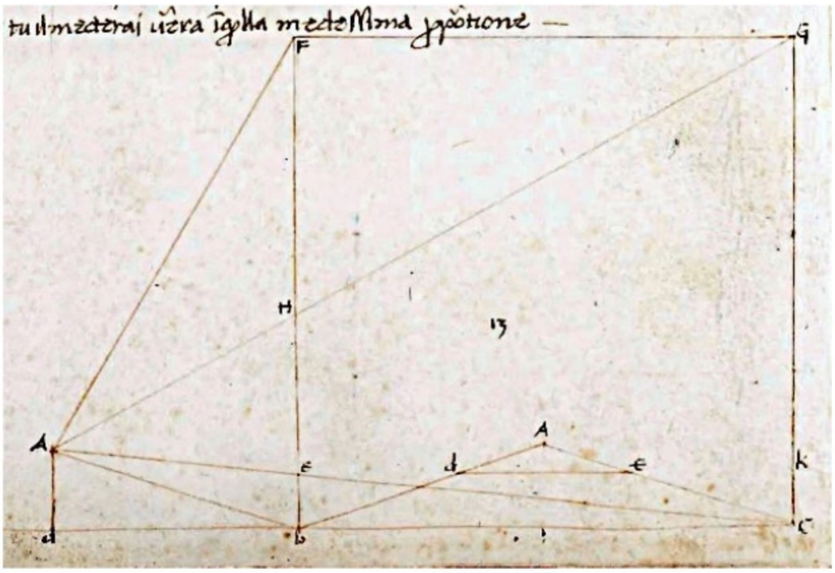

Figure 1: Piero Della Francesca, *De Prospectiva Pingendi*, Book I, Fig. 13. Perspective of a cubic space: geometric process. Original Diagram [2].

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Isoptic surfaces of segments in $\mathbf{S}^2 \times \mathbb{R}$ and $\mathbf{H}^2 \times \mathbb{R}$ geometries

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In this work we examine the isoptic surfaces of line segments in the $\mathbf{S}^2 \times \mathbb{R}$ and $\mathbf{H}^2 \times \mathbb{R}$ geometries, which belong to the 8 Thurston geometries. Based on the procedure first described in [1], we are able to give the isoptic surface of any segment implicitly. We rely heavily on the calculations published in [2]. As a special case, we examine the Thales spheres, called Thaloids, in both geometries. In our work we use the projective model of $\mathbf{S}^2 \times \mathbb{R}$ and $\mathbf{H}^2 \times \mathbb{R}$ described by E. Molnár.

Key words: isoptic surface, projective geometry, Thurson geometries, isometry

MSC 2010: 53A20, 53A35, 52C35, 53B20

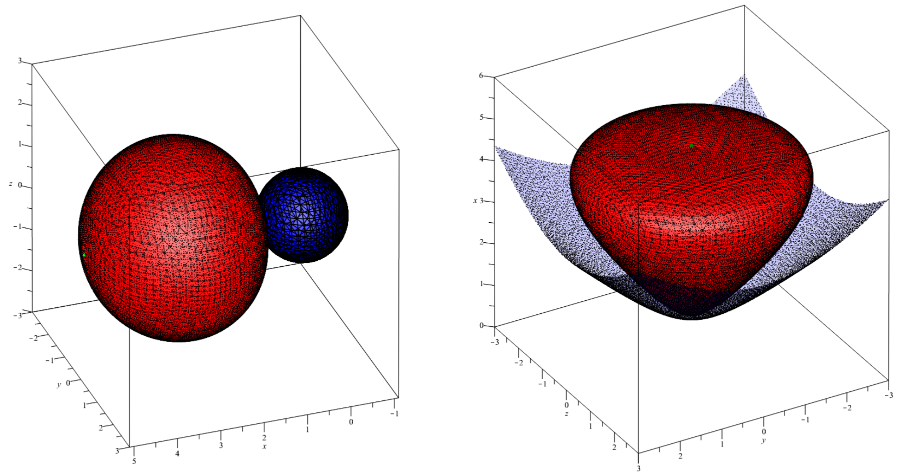

Figure 1: Thaloid of the $A_1(1,1,0,0)$ and $A_2(1,5,0,0)$ segment in $\mathbf{S}^2 \times \mathbb{R}$ (left) and $\mathbf{H}^2 \times \mathbb{R}$ (right) geometries.

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Weierstrass representation of lightlike surfaces in Lorentz-Minkowski 4-space

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We present a Weierstrass-type representation formula which locally represents every regular two-dimensional lightlike surface in Lorentz-Minkowski 4-space by three dual functions ρ, f, g. The formula generalizes the representation for regular lightlike surfaces in 3-space. We also give necessary and sufficient conditions on the dual functions ρ, f, g for the surface to be minimal, ruled or l-minimal.

Key words: lightlike surfaces, Lorentz-Minkowski 4-space, Weierstrass representation formula

MSC 2010: 53A35, 53B30

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AI supported innovative math assessment

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Artificial intelligence (AI) rapid development cannot be overlooked by mathematics teachers. We can consider AI as a thread to fairness and reliability of assessment but also take advantage the opportunity for changing and introduce meaningful, nonroutine and challenging assessment tasks.

Inquiry-based and problem-solving tasks can be valuable part of the student assessment in mathematics. We introduced a new form of problem-solving task that students prepare at home and then report in form of an essay and oral presentation. Innovative element was to advice students to partner with a chatbot (e.g. ChatGPT) in problem investigation and critically analyse results.

The students were given a description of their individualized problem-solving exercises. The typical exercise consists of theoretical background research and then based on that students need to solve (not too difficult) problem.

The learning outcomes linked to that task include problem solving, structuring mathematical text, preparing graphical representations if appropriate, using tools for mathematical text editing (Latex) as well as correctly listing and correctly using references. Additionally, students reported the results of their interaction with the chatbot, detected mistakes, and further researched the mathematical topic to solve a problem. Students were also advised to carefully use Croatian standard mathematical terminology and take care of correct expressions of definitions and theorems. At the end, students are supposed to analyse the problem solution(s) as well as critically evaluate of their interaction with the chatbot. We have also collected feedback from students related about this exercise. Students feedback showed that they were motivated by use of AI in mathematical problem-solving and essay writing, and they noted many advantages and disadvantages of use of AI in that assessment task.

Finally, since students' essays were assessed by the scoring rubric, we analysed what type of problem-solving topics were successfully investigated and presented by AI support.

Key words: math assessment, artificial intelligence, problem-solving task

MSC 2010: 97-06

On translation curves and geodesics in Sol_1^4 space

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A translation curve in a Thurston space is a curve such that for a given unit vector at the origin translation of this vector is tangent to the curve in every point of the curve. In most Thurston spaces translation curves coincide with geodesic lines. However, this does not hold for Thurston spaces equipped with a twisted product. In these spaces translation curves seem more intuitive and simpler than geodesics.

In this talk we consider translation curves and geodesics in Sol_1^4 space and explain the curvature properties of translation curves.

Key words: geodesic; translation curve, solvable Lie group, Sol_1^4 space

MSC 2010: 53C30, 53B20, 53C22

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Involute and evolute of partially null curve in 4-dimensional Lorentz-Minkowski space

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Involutes and evolutes of curves in Euclidean plane were introduced already in 1670's by Huygens, in relation to his work on pendular motion and the isochronous pendulum [3]. The definitions of involutes and evolutes, used in this presentation, are inspired by the analogous definitions for Euclidean case [5]. The orthogonal trajectories of the first tangents of the curve c are called the involutes of the curve c. We say that a curve c^* is an evolute of curve c if c is the involute of c^*.

Involutes and evolutes in n-dimensional Euclidean space and simply isotropic space were investigated in [1] and [4], respectively. In the 4-dimensional Lorentz-Minkowski space, the involute and evolute curves of a spacelike curve with non-null normals have been investigated in [8, 9]. Involute and an evolving involute of order k of a null Cartan curve in n-dimensional Minkowski space have been investigated in [6].

In [7] we analyzed involutes of pseudo-null curves, that is, spacelike curves with null principal normals. We continued our research and investigated properties of involute and evolute of partially-null curve, that is, spacelike curve whose first binormal vector is null in 4-dimensional Lorentz-Minkowski space [2] and those results are to be presented.

Key words: partially-null curve, Minkowski space, evolute, involute

MSC 2010: 53A10, 53B30

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Isometries and their square roots – from the Euclidean plane to various normed spaces and back

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One of the most important topics in the Euclidean geometry from the earliest times has been the study of distance-preserving transformations. These transformations automatically preserve other geometric quantities such as angle and area. A distance-preserving transformation is called an isometry and it can be defined not only on the Euclidean plane, but also on any metric space, and especially on any normed space. The study of isometries between normed spaces has been a particularly fruitful and active research topic. It is clear that the square of any isometry is again an isometry. The aim of this talk is to consider the converse of this fact in various normed spaces and also in the Euclidean plane. More specifically, it will be considered whether and under what conditions it is true that a given isometry is the square of some isometry (with respect to the same norm). This talk is motivated by [1].

Key words: isometry, Euclidean plane, normed space

MSC 2010: 46B04, 51M04

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On the geometry of spherical trochoids

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This talk is based on an unpublished manuscript by our former colleague and teacher Walther Jank (1939–2016). He was a student and dedicated follower of Walter Wunderlich and his work on kinematics. In 2004, he gave one of his last talks at the geometry conference in Vorau. With this reminiscence, we want to pay a tribute to Walther Jank and recall some geometry that is in danger of getting lost.

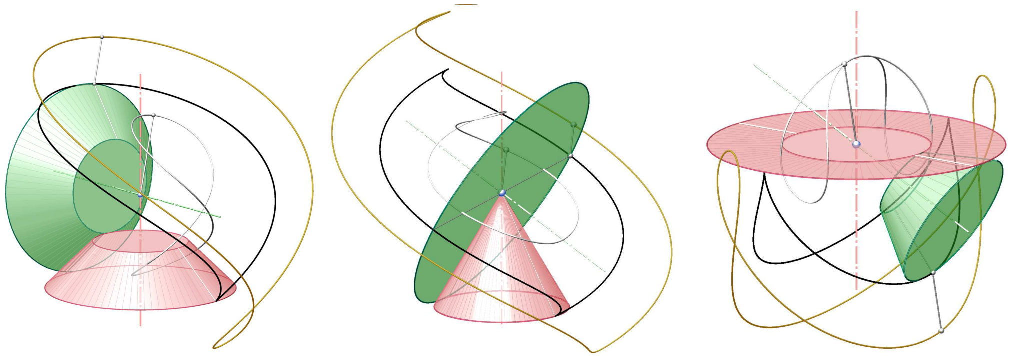

Figure 1: Spherical trochoids generated as point paths of rolling cones.

Spherical trochoids are orbits of points on a sphere undergoing a spatial rigid body motion that is the superposition of two rotations about intersecting axes (cf. [3, 7]). Since the axodes of these motions are cones of revolution, spherical trochoids can also be generated as paths of points rigidly attached to a cone of revolution that rolls (without gliding) along another cone of revolution (cf. Fig. 1).

Equipped with some knowledge about planar trochoids (see [9]) and planar trochoids of higher order (as defined in [8]), the orthogonal projections of spherical trochoids onto a triplet of mutually orthogonal planes (top view, front view, right-side view) are studied (cf. Fig. 2).

These projections turn out to be trochoids of order three, *i.e.*, curves generated by the superposition of more than two rotations (cf. [1, 8]), while general oblique projections result in trochoids of even higher degree, see [1]. Proofs are based on synthetic reasonings and do not use any computations.

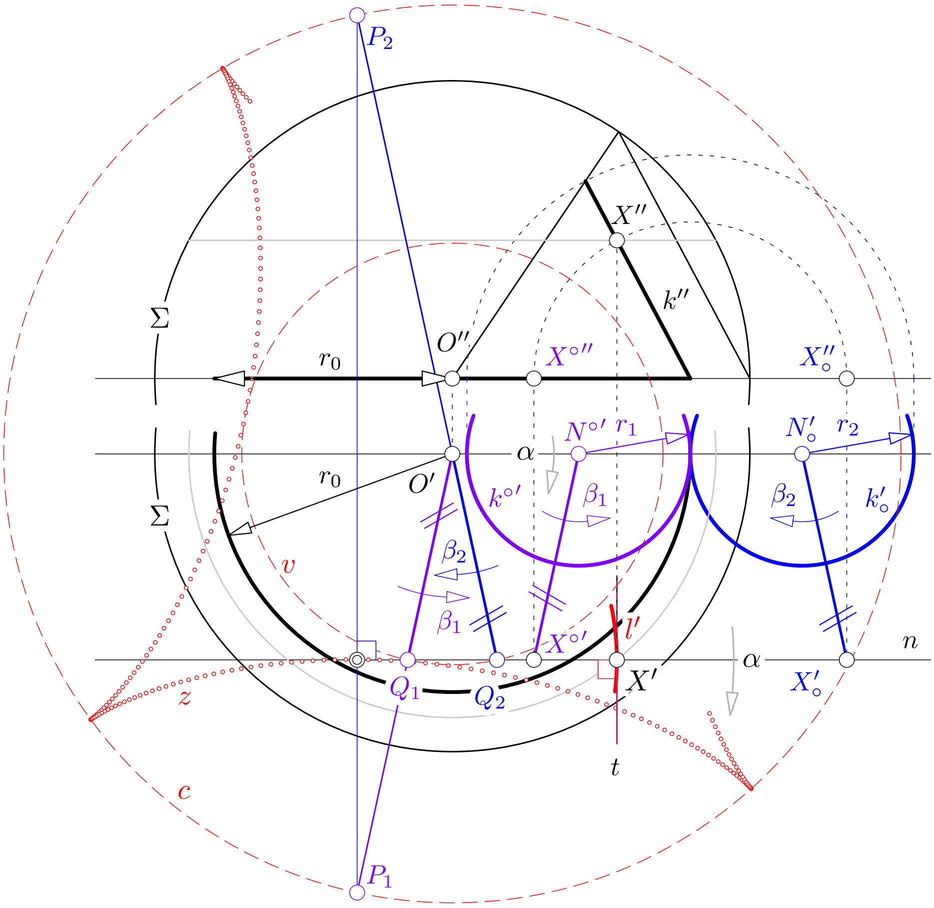

Figure 2: The top-view l' of a spherical trochoid is the involute of a hypocycloid z.

Special shapes of these trochoids show up for special choices of the spherical radii of the rolling circles. Further, the principal views of spherical trochoids allow for simple kinematic and geometric generations. Finally, it turns out that some spherical trochoids map to curves of constant width (cf. Fig. 3). This leads to a question raised by W. Jank: Can all planar curves of constant width (analytic and closed) be considered as projections of spherical trochoids? In fact, there is some work on that topic (cf. [2, 4, 5, 6]) and the approaches towards planar and closed (not necessarily convex) analytic curves of constant width naturally differ from the constructive approach.

Key words: spherical trochoid, rolling, evolute, involute, curve of constant width

MSC 2020: 53A17, 51N05, 14H45

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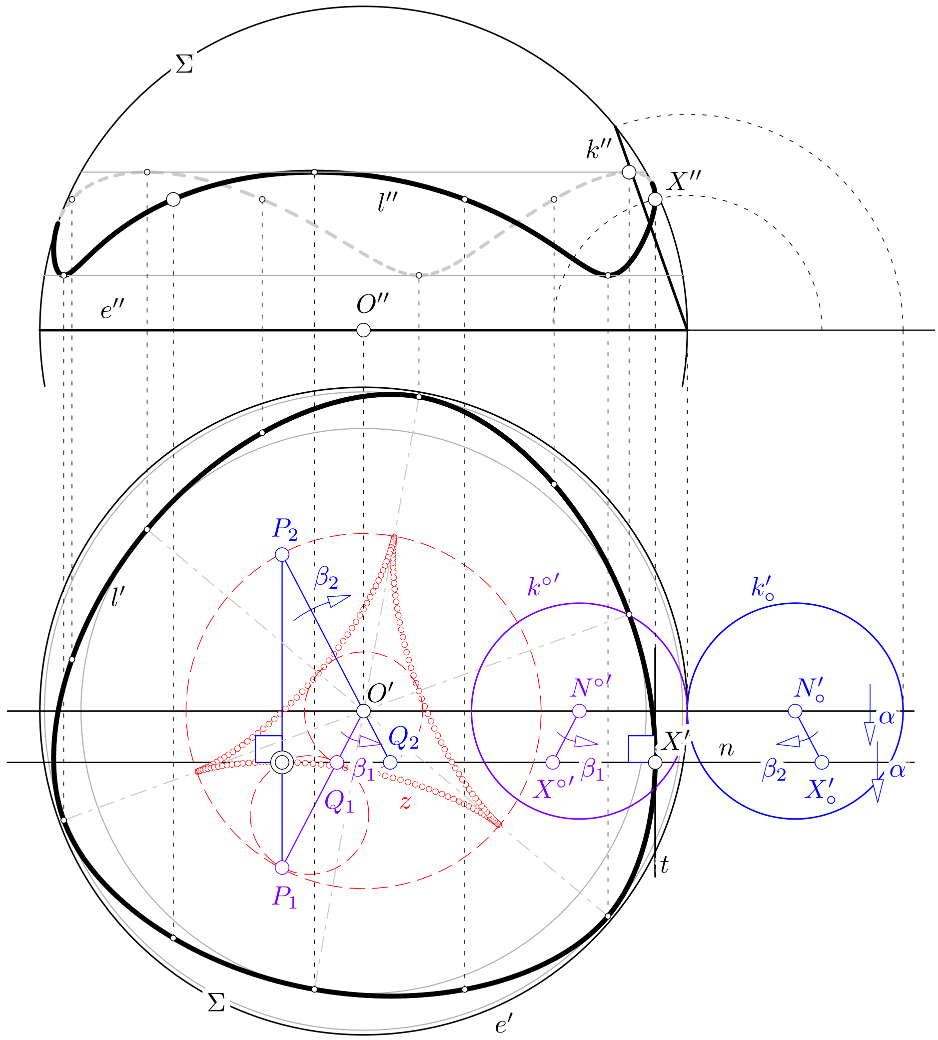

Figure 3: The top-view $l'$of a spherical trochoid may even be a closed and algebraic curve of constant width.

Bisectors of conics in the isotropic plane

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In this presentation we introduce the concept of a bisector of two curves in an isotropic plane. We generalize the concept of the bisector of straight lines to the bisector of curves of higher degrees:
Let the curves $\mathcal{A}$ and $\mathcal{B}$ of degrees m and n, respectively, be given. Each isotropic line (line through the absolute point F) intersects $\mathcal{A}$ in m points A_i, $i = 1 \dots m$, and $\mathcal{B}$ in n points B_j, $j = 1, \dots n$. Let T_{ij} be harmonic conjugates of F with respect to A_i, B_j, i.e. $\mathrm{H}(A_i, B_j, F, T_{ij})$. The locus $\mathcal{K}$ of all points T_{ij} for all isotropic lines is called the *bisector curve* of $\mathcal{A}$ and $\mathcal{B}$.

We study the bisectors of conics and classify them according to their type of circularity. In general the bisector of conics is a quartic, while in the case when both conics are circular, its degree decreases.

Key words: isotropic plane, conic, bisector, circular curve

MSC 2010: 51N25

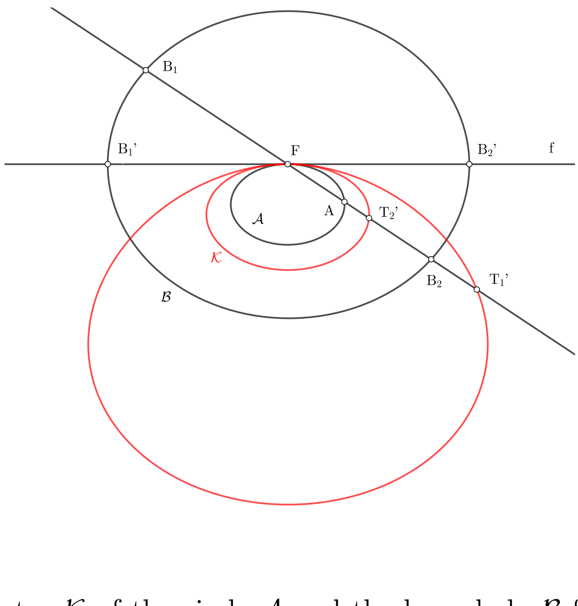

Figure 1: The bisector $\mathcal{K}$ of the circle $\mathcal{A}$ and the hyperbola $\mathcal{B}$ from the projective point of view.

Hilbert's irreducibility, modular forms, and computation of certain Galois groups

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Joint work with Goran Muić

We consider congruence subgroups $\Gamma_0(N)$, $N \geq 1$, and the corresponding compact Riemann surface $X_0(N)$ which we initially consider as a complex irreducible smooth projective curve. The $\mathbb{Q}$–structure on $X_0(N)$ is defined in a standard way using j–function i.e., the field of rational functions over $\mathbb{Q}$ on $X_0(N)$ is given by $\mathbb{Q}(X_0(N)) = \mathbb{Q}(j, j(N\cdot))$ where $j(N\cdot)$ is the meromorphic function $z \longmapsto j(Nz)$ on the upper half plane.

The paper [3] constructs various models over $\mathbb{C}$ of $X_0(N)$ complementing previous works on this subject. In our newest paper [1] we describe another application of [2] and [3] where we justify why it is interesting to study plane models of $X_0(N)$ of various type not necessarily of smallest possible degree or one with smallest possible coefficients. It is known that spaces of modular forms $S_m(\Gamma_0(N))$ and $M_m(\Gamma_0(N))$, for $m \geq 2$ even, have basis consisting of forms with integral q–expansions. So, if we take that f, g, h are linearly independent modular forms with rational q–expansions for $\Gamma_0(N)$, we can construct an irreducible over $\mathbb{Z}$ homogeneous polynomial with integral coefficients $P_{f,g,h}$ such that $P_{f,g,h}(f, g, h) = 0$ in $\mathbb{Q}(X_0(N))$. Let $Q_{f,g,h}$ be its dehomogenization with respect to the first variable. Again, we obtain an irreducible over $\mathbb{Z}$ polynomial with integral coefficients. By means of Hilbert's irreducibility theorem (see [4]), we obtain a family of irreducible over $\mathbb{Z}$ polynomials with integral coefficients $Q_{f,g,h}(\lambda, \cdot)$, where λ ranges over a certain thin set. In this way we obtain a family of number fields determined as splitting fields of these polynomials all of which have the same Galois group $G_{f,g,h}$ which is the Galois group of the splitting field of $Q_{f,g,h}(g/f, \cdot)$ over $\mathbb{Q}(g/f)$. The goal is to study these objects as an application of the theory developed in [2] and [3].

Key words: modular forms, modular curves, birational equivalence, Galois groups, Hilbert's irreducibility theorem

MSC 2010: 11F11, 11F23

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Feuerbach point and Feuerbach line of a triangle in the isotropic plane

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In this talk, we study the Feuerbach point and the Feuerbach line of a triangle in the isotropic plane. We present statements about these concepts in the isotropic plane, and consider a number of properties of the introduced concepts. We explore relationships between the Feuerbach point and the Feuerbach line and some other elements related to the given triangle. We also investigate relationships between the Euclidean and the isotropic case.

Key words: isotropic plane, Feuerbach point, Feuerbach line

MSC 2010: 51N25

Geometry of the roundabouts

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A modern roundabout can be defined as an intersection with raised central island and a circulatory roadway at which yield-at-entry rule is applied. Due to the yield-at-entry rule, an appropriate sight distance must be ensured to enable a driver entering the roundabout safely, to note on time the position of the vehicle with the right of way and to have an obstructed view of the opposite exit. In this talk we will present some of the geometry of the roundabouts referring to the as paraboloids concerning the sight distance as development of a geometric model that aids in the roundabouts central island design by determining the maximum size of the visibility obstacle.

Key words: roundabouts, sight distance, paraboloid

MSC 2010: 14J26, 51N20

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Alternating sign matrices and Dyck paths

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Alternating sign matrices are matrices whose non-zero elements alternate in sign, and that sum to 1 per each row and column. In this paper, we extend the notion of permutation pattern to these matrices. We study a family of alternating sign matrices with permutation pattern avoidance and a constraint on relative positions of 1s among neighboring rows. Refined enumerations of these matrices with respect to the special element and with respect to position of $1s$ in the first row are provided. We also study relations to the internal triangles, which apper within convex polygon triangulations.

Key words: Alternating sign matrices, Dyck paths, Narayana numbers, internal triangles

MSC 2010: 15B35, O5A05.

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Explicit methods with modular curves and Weierstrass points

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Joint work with Goran Muić

In this talk we describe recent joint work [1] with Goran Muić on determination of ordinary and generalized Weierstrass points on a complex smooth projective algebraic curve. An algorithm has been developed in SAGE that works for all curves of type $X_0(N)$, of the genus $g \geq 3$, that are not hyperelliptic [2].

Key words: modular forms, algebraic curves, uniformization theory, Weierstrass points

MSC 2010: 11E70, 22E50

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Hyperbolic crystal geometry, on the 200th anniversary of János Bolyai's absolute geometry

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A compact manifold of constant curvature ($K = 0$ Euclidean $\mathbf{E}^3$, $K > 0$ spherical $\mathbf{S}^3$, $K < 0$ hyperbolic $\mathbf{H}^3$) is called space form. This concept can naturally be extended to any other space X ($\mathbf{S}^2 \times \mathbf{R}$, $\mathbf{H}^2 \times \mathbf{R}$, $\widetilde{\mathbf{SL}_2\mathbf{R}}$, **Nil**, **Sol**) of the 8 Thurston's (simply connected homogeneous Riemann) 3-geometries. Thus, we look for a fixed-point-free isometry group G acting on X with compact fundamental polyhedron $F = X/G$, endowed by appropriate (tricky) face pairing identifications.

It turned out that the previous (1984-88) initiative of the first (* presenting) author, constructing new hyperbolic space forms (football manifolds) with István Prok's computer implementations and Jenö Szirmai's numerical computations, have got applications in crystallography, e.g. as fullerenes, nanotubes. Nowadays we (also in international cooperation, see the selected list of References and citations therein) found further space forms also in other Thurston spaces. Furthermore, extremum problems and other possible applications, as infinite series $Cw(2z,\ 3 \leq z \in N$ odd number) of hyperbolic nanotubes with any z-rotational symmetry are foreseen.

Maybe, our experience space in small size can be non-Euclidean as well !!!

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On embeddings of projective algebraic curves in projective spaces

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We use certain explicit results on modular forms via uniformization theory to obtain embeddings of modular curves and more generally of compact Riemann surfaces attached to Fuchsian groups of the first kind in certain projective spaces. We obtain families of embeddings which vary smoothly with respect to a parameter in the upper–half plane.

Key words: modular forms, algebraic curves, compact Riemann surfaces, uniformization theory

MSC 2010: 11E70, 22E50

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A Miquel-Steiner transformation

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Each complete quadrilateral in the Euclidean plane has three Miquel points which are the common points of three quadruples of circumcircles of the quadrilateral's subtriangles. Any triangle $\Delta = ABC$ together with an arbitrarily chosen point Z not on a triangle side also defines a complete quadrilateral, and thus, this pivot point defines three Miquel points with respect to Δ (see Figure 1, left).

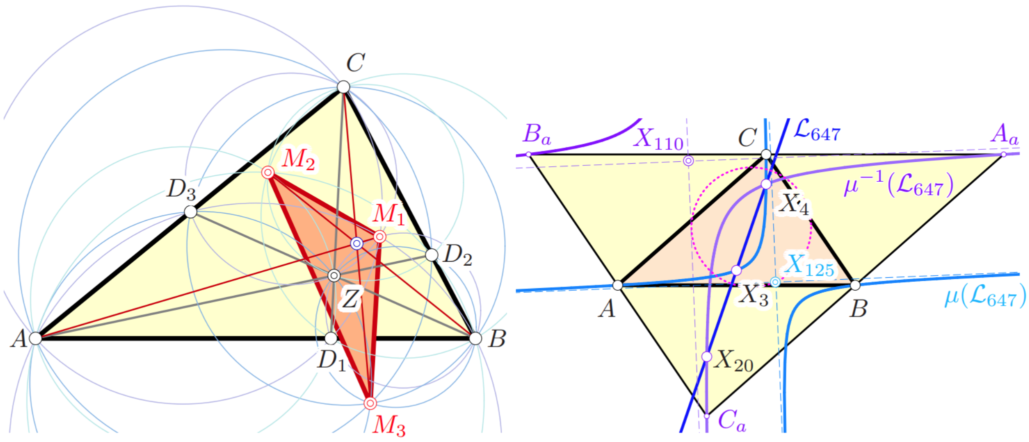

Figure 1: Left: Miquel points of a point Z with respect to a triangle $\Delta = ABC$ and the Miquel perspector P.
Right: The Euler line and its μ-image and μ^{-1}-image.

It turns out that these three Miquel points form a triangle Δ_M which is perspective with the base triangle. The mapping μ that assigns to the pivot point Z the uniquely defined perspector P is a quadratic Cremona transformation and shall be called *Miquel-Steiner transformation*.

Its exceptional set consists of the side lines of Δ's anticomplementary triangle Δ_a. Unlike many other known quadratic Cremona transformations, the Miquel-Steiner transformation is not involutive. The exceptional set of the inverse mapping μ^{-1} consists of the side lines of Δ. We shall study the action of the Miquel transformation at hand of some examples, e.g., the images of the Euler line (see Figure 1, right), the antiorthic axis, the incircle, and the circumcircle.

In some case this leads to surprisingly simple constructions of tangents at double points of image curves. Further, right triangles play a special role.

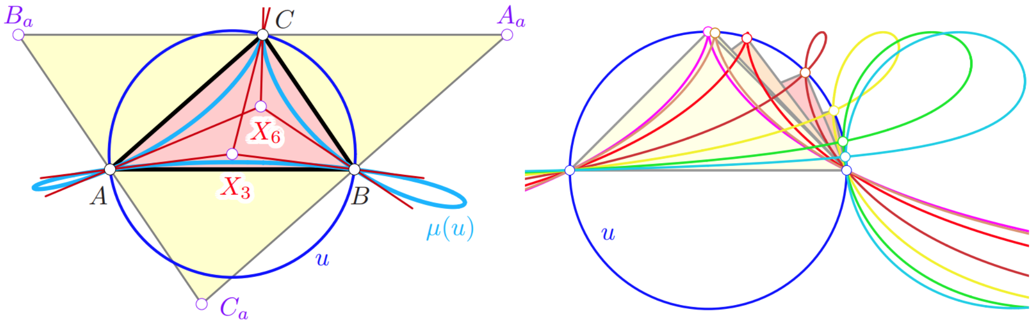

Figure 2: Left: The Miquel transform of the circumcircle u is a quartic with three ordinary double points at the vertices of Δ. The tangents at the double points are the Cevians of X_3 and X_6.
Right: A sequence of right triangles with ratios of cathetus's lengths 1:1, 20:21, 3:4, 5:12, 9:40, 19:180, 41:840 and the corresponding cubic curves as μ-images of the circumcircle u.

Key words: Miquel points, quadrilateral, triangle, quadratic Cremona transformation

MSC 2020: 14A25, 51N15

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Constrained topological networks for advanced 3D scanning analysis

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Point clouds often contain a vast amount of redundant information, which makes it challenging to process the points efficiently. Working directly with each point can be a computationally expensive and time-consuming task due to the substantial number of individual points, which can range from millions to billions. Reducing the redundant information and extracting meaningful features from the point cloud is necessary to enable efficient point cloud processing algorithms

The goal of this work is to introduce and present an efficient and flexible data reduction algorithm for 3D scanning applications. It utilizes constrained topological networks to handle the vast amount of data existing in point clouds and has the potential to be employed in various applications, such as geometric inspection, reverse engineering, and shape optimization, making it highly valuable in the domain of mechanical engineering.

Drawing upon the recognition that a significant portion of entities encountered in mechanical engineering or architecture, for example, can be simplified using fundamental geometric elements such as planes and cylinders, this research contributes the most by presenting an exhaustive framework for minimizing the density of intersecting point clouds through the application of constrained topological networks. By tackling the obstacles inherent in point cloud analysis and processing, this study propels the progress of 3D scanning methodologies, specifically in the realm of mechanical engineering, empowering the optimized and impactful utilization of point cloud data with enhanced efficiency and effectiveness.

Key words: point-cloud, topological networks, recognition, graph matching, 3D scanning

Ivory's theorem and self-adjoint mappings

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Ivory's Theorem deals with confocal surfaces and is valid in Euclidean, hyperbolic, and elliptic spaces of all dimensions. The three-dimensional version states that in each curvilinear box formed by confocal conics the spatial diagonals are of equal lengths (see Figure 1). As stressed in [1] for Euclidean and in [2] for hyperbolic spaces, Ivory's theorem is closely related to self-adjoint mappings. We focus on pseudo-Euclidean spaces $\mathbb{M}^n_{n-p}$ and prove a converse: For all self-adjoint affine transformations in $\mathbb{M}^n_{n-p}$ a generalized version of Ivory's theorem is valid which in almost all cases leads directly to confocal quadrics.

Key words: Ivory's theorem, confocal quadrics, self-adjoint mappings, pseudo-Euclidean space

MSC 2010: 51N10, 51N25

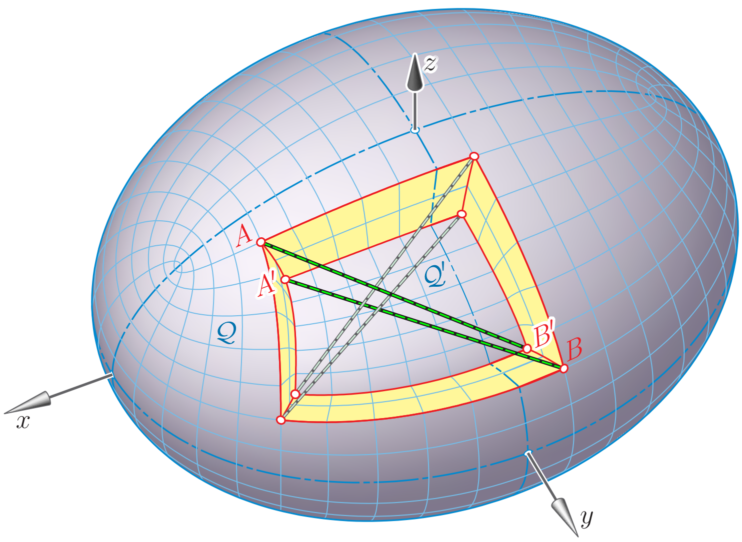

Figure 1: Ivory box in $\mathbb{E}^3$ bounded by three pairs of confocal central quadrics. The four spatial diagonals are of equal lengths, in particular $\overline{AB'} = \overline{A'B}$.

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Computed Cake 4.0

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Rethinking new sustainable building materials starts with using local products and combining them in unexpected ways. The challenging questions here are: Can we think about building materials in the context of our daily activities and manufacturing processes? How can we learn from cooking and combine it with parametric design and digital fabrication processes such as 3D printing, vacuuming, laser cutting, CNC milling, etc.? How can we harness the creative possibilities of architecture for design?

In this presentation, an ongoing project with architecture students from Graz University of Technology will be presented (Fig. 1). The main goal of the project is to combine geometry, theory and methods of parametric design with a cooking project. The geometry of cakes is developed digitally, corresponding tools are designed and the "taste" is translated into a corresponding artwork.

Key words: digital fabrication, parametric design

MSC 2010: 51N99

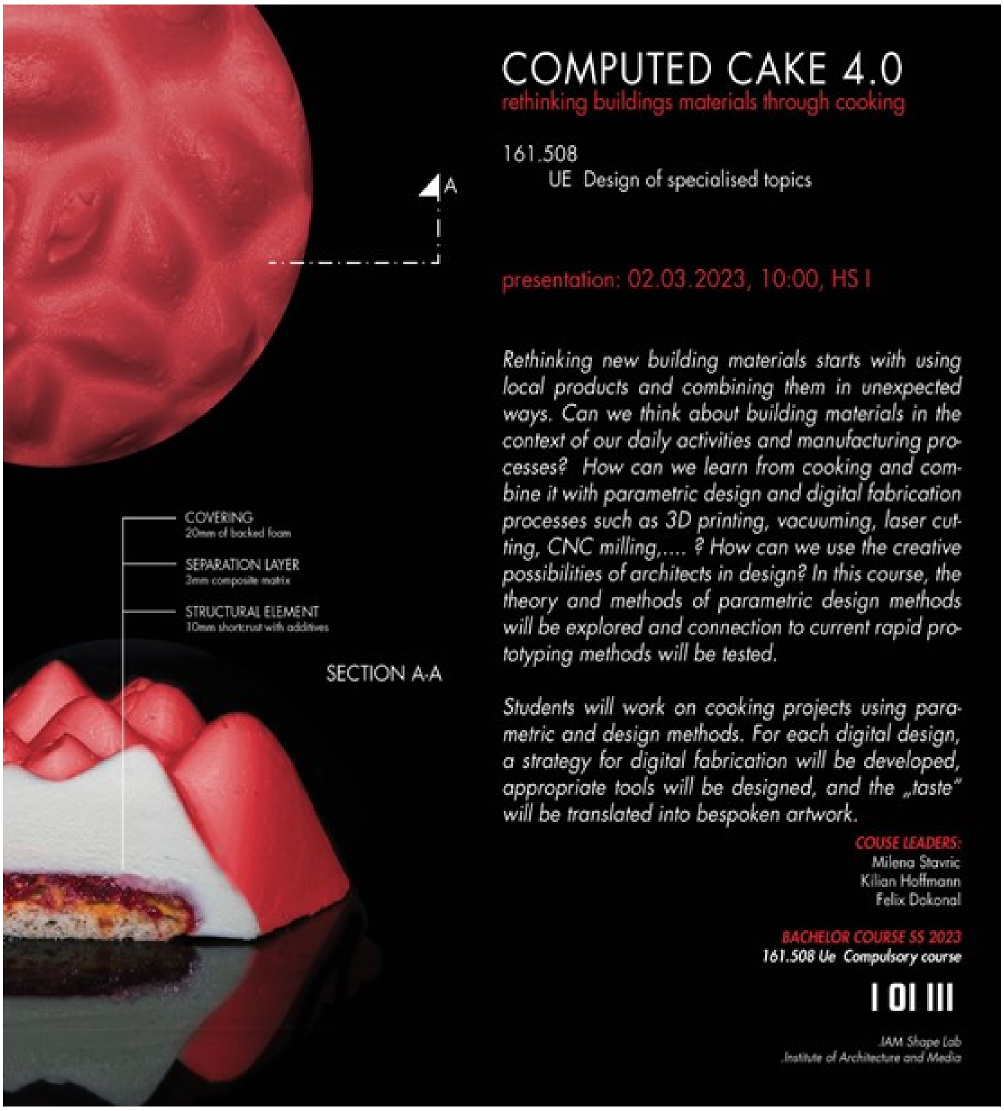

Figure 1: Poster of the project.

Symmetry

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Symmetry is a general concept with slightly different meaning as used in different sciences:

- in geometry – the property by which the elements of a figure or object reflect each other across a line (axis of symmetry) or surface;
- in mathematics – a basic property of many mathematical objects in various mathematical theories – calculus, linear algebra, vector calculus, abstract group theory, Galois theory, set theory, topology, combinatorics and statistics;
- in biology – the orderly repetition of parts of an animal or a plant;
- in chemistry – an essential undergirding of all specific interactions between molecules and orderly arrangements of particles in molecules and atoms;
- in crystallography – the orderly arrangements of atoms found in crystalline solids;
- in physics – a concept of balance described by many fundamental physical laws;
- in psychology and neural science – a special sensitivity to the perception of reflection symmetry in humans as a general response to various types of regularities;
- in social sciences – the symmetrical nature of human interaction, often including asymmetrical balance in a variety of contexts;
- in aesthetics – a complex concept of perception of beauty, harmony and disharmony;
- in visual art and architecture – the classical basic principle of harmony, accord and structural framework, transformed in modern art and architecture as "wings and balance of masses";
- in music – an important consideration in the formation of scales and chords, invariance and permutations;
- in literature – the symmetrical structure of story framework, palindrome words.

Symmetry in nature underlies one of the most fundamental concepts of beauty. It connotes balance, order, and thus, to some extent, a type of divine principle.

Valid symmetry operations are those that can be performed without changing the appearance of an object. The number, type and variety of such operations depend on the geometry of the object to which the operations are applied.

Some interesting applications of the concept of symmetry will be presented, that are leading to the elegant solutions of various difficult fundamental problems.

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Brocard seen with Miquel's eyes

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The elementary geometric Miquel theorem concerns a triangle $\triangle ABC$ and points R, S, T on its sides, and it states that the circles $\text{O}ART$, $\text{O}BRS$, $\text{O}CST$ have a common point M, the Miquel point to these givens. For M there exists a two-parametric set of possibilities, such that there exists a one-parameter set of point triplets R, S, T to a given point M. Choosing R, S, T in special ways one receives the so-called *beermat theorem*, the *Brocard theorems*, and the *Simson-Wallace theorem* as special cases of Miquel's theorem. Hereby facts connected with Brocard's theorem follow from properties of Miquel's theorem. Furthermore, if the points R, S, T run through the sides of $\triangle$ such that the ratios $R(ARB)$, $R(BST)$, $R(CTA)$ are equal, then the corresponding Miquel points M run through the circumcircle of the triangle with the Brocard points and the circumcenter of $\triangle$ as vertices. Besides these three remarkable points of $\triangle$ this circle contains several other triangle centers, see [1].

Key words: Brocard points, Miquel's theorem

MSC 2010: 51M04

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Geometry in the “Digital Twin” concept

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The concept of a Digital Twin has gained considerable attention in recent years as a powerful tool for representing and simulating physical objects or systems in the digital world. While the term “digital twin” covers several aspects, this presentation focuses specifically on the geometric component of digital twins. Geometric representation is a fundamental component of Digital Twin and enables accurate modeling and visualization of physical objects and their behavior in the virtual environment.

Geometric modeling techniques such as computer-aided design and 3D scanning technologies play an important role in creating the geometric representation of physical objects for Digital Twins. CAD models provide a three-dimensional blueprint for digital replicas and capture the essential geometric details and spatial relationships. These models serve as the basis for simulating the behavior and interactions of their corresponding physical counterparts.

3D scanning technologies contribute to the creation of digital twins by capturing the geometry of existing objects or environments. Techniques such as laser scanning or photogrammetry capture precise measurements and point cloud data that enable the creation of highly accurate digital replicas. The geometric information obtained during 3D scanning serves as the basis for developing virtual representations that closely resemble physical reality. Although laser scanning is very precise, the geometric information and coherence of the components is usually lost and it often requires a “human” geometric eye to correctly identify the relationships.

The geometric aspect of Digital Twins goes beyond static models to include dynamic simulations. Through the use of parametric and algorithmic modeling based on geometric and physical principles, virtual models can mimic the real-world behavior of physical models to some degree. Such an approach enables optimization, prediction, and analysis of the entire system.

In addition, the geometric component of Digital Twins enables immersive visualization and interaction with the digital replicas. Virtual reality (VR) and augmented reality (AR) technologies allow users to explore and manipulate geometric models, enhancing their understanding and engagement with the physical objects they represent. This interactive experience bridges the gap between the digital and physical worlds, opening up new opportunities for collaboration, training, and experimentation.

Key words: digital twin, geometric representation, parametric modelling

MSC 2010: 51N99, 51P05

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Stationary distances between spatial circles

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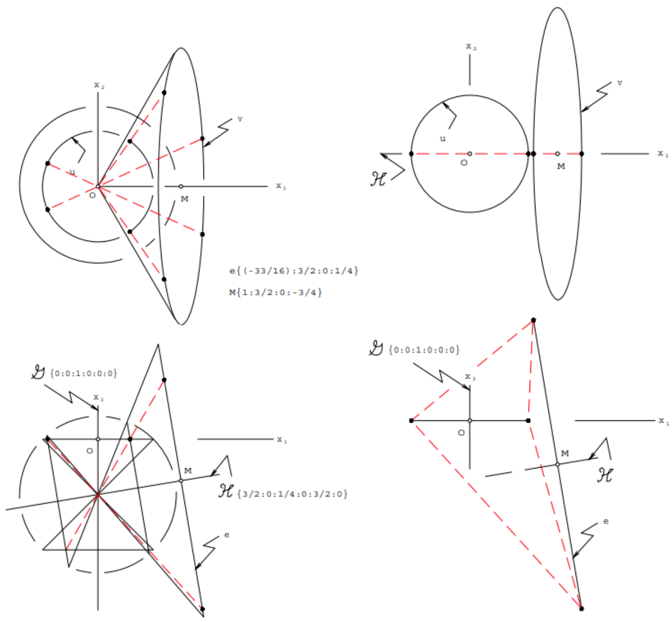
This is about connections via elements in the congruence of lines that are locally normal to and intersect spatial circles. Observe that these lines form a pencil of right cones whose apices lie on the circle axis. Four specific instances are discussed; point-to-circle, plane-to-circle, line-to-circle and circle-to-circle. The first two admit two connections, the third up to four while two circles can be connected normally in up to eight ways. Shortest and longest connections are among the eight. Complex solutions appear in many configurations of line-to-circle and circle-to-circle. Experience with this issue began in 2004 with [1]. There it lay until we asked how arrangement of things to be connected affects real solutions and their symmetry.

Key words: geometry, circles, line, segment, connection

MSC 2010: 14-01, 14-06



(a) Four Joining Lines Symmetric to the Plane of Circle Axes

(b) Four Joining lines on Plane of Circles Axes

Figure 1: Eight connections between circles on concentric spheres.

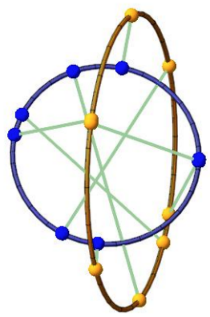


Figure 2: Gfrerer shows eight connections between circles with skew axes.

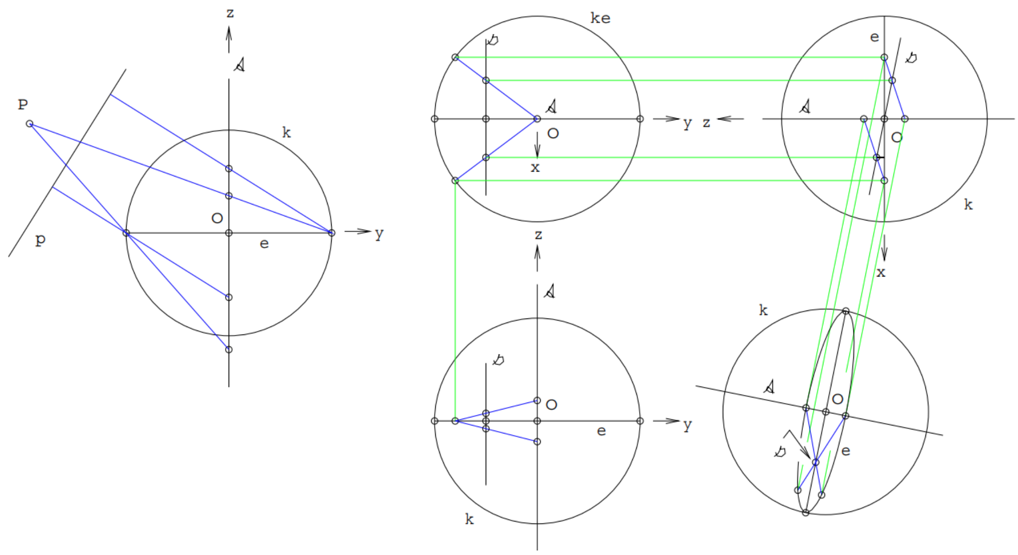


Figure 3: Two connections between point or plane and circle four between line and circle but peculiar to coplanar common normal from line to circle.

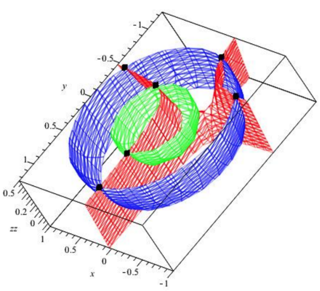


Figure 4: More general line-to-circle

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Posters

Geometry and graphics in architecture through the eye of AI

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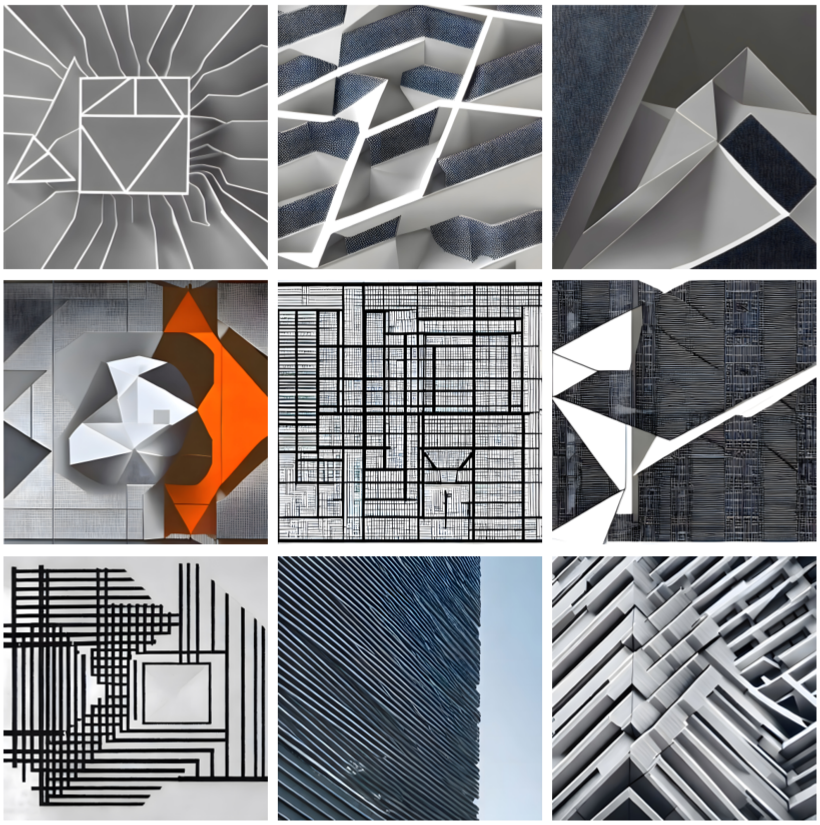
Artificial Intelligence, if such a thing can truly exist, is a tool used by many to generate all sorts of visual representations based on elaborate prompting and filtering. Different free and open AI tools like DALL·E, Stable Diffusion, PlaygroundAI etc are available with different possibilities and rendering styles. By using prompts like "Architectural Graphics", "Geometry in Architecture" and alike, a pool of square images were generated using Stable Diffusion from which selected are those that border abstract painting and architecture, and display the most regular structure, clean surfaces, and visually attractive graphism.



When 5, 7, 9, 10 and π meet descriptive geometry

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Making use of the concept: approximate / construct /apply, we demonstrate moments in the teaching of descriptive geometry in which one needs to reach out for approximations in the process of construction (transcendent number π, regular heptagons and nonagons.) Classical constructions that include golden number φ are recalled and applied as well.

Key words: approximations, descriptive geometry, geometry education

MSC 2010: 97G80

Associated surfaces of a maximal/minimal surface in Lorentz-Minkowski 3-space and their exponential map

Davor Devald

We give a result on the relationship between the exponential maps of associated surfaces S_θ, $\theta \in \mathbb{R}$ of a maximal/minimal surface S in $\mathbb{M}^3$. We also find an explicit formula for the geodesics and the exponential map of such surfaces by using complexification of the local coordinates (u, v). For a lightlike surface, we construct the analogous family of associated surfaces and find the ODE's for it's geodesics by using dual functions.

Key words: associated family, exponential map, geodesics

MSC 2010: 53A35, 53B30

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Sweet division problems: Chocolate bars and honeycomb strips

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We consider two division problems on narrow strips of square and hexagonal lattices. In both cases we compute the bivariate enumerating sequences and the corresponding generating functions, which allowed us to determine the asymptotic behavior of the total number of such subdivisions and the expected number of parts. In the hexagonal case, we find a number of new combinatorial interpretations of the Fibonacci numbers and find combinatorial proofs of some Fibonacci related identities.

Key words: tilings, divisions, rectangular grid, hexagonal grid

MSC 2010: 52C20, 05B45

Orthopoles related to a complete quadrangle

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Joint work with Vladimir Volenec

We study a complete quadrangle $ABCD$ in the Euclidean plane that has a rectangular hyperbola $\mathcal{H}$ circumscribed to it. The approach is based on the rectangular coordinates and we prove the following main result:
Let $ABCD$ be a complete quadrangle and l_a, l_b, l_c, l_d mutually parallel lines through the circumcenters of BCD, ACD, ABD, ABC, respectively. Orthopoles of the lines l_a, l_b, l_c, l_d with respect to the triangles BCD, ACD, ABD, ABC lie on a line which passes through the center of the rectangular hyperbola $\mathcal{H}$ circumscribed to $ABCD$, and it is antiparallel to the given lines with respect to the axes of the hyperbola $\mathcal{H}$.

Key words: complete quadrangle, orthopole, rectangular hyperbola, rectangular coordinates

MSC 2010: 51N20

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